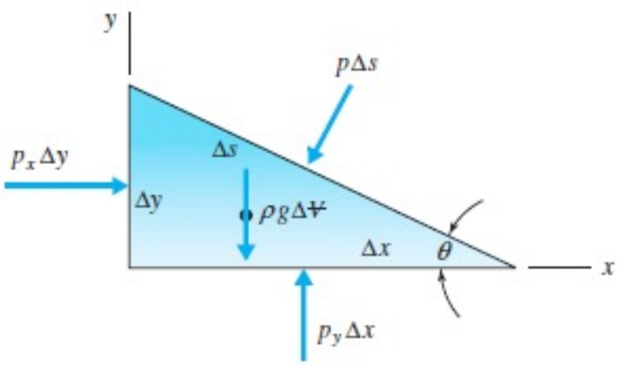
Fluid Statics

Introduction

- **Fluid Statics**: Study of fluids with no relative motion between fluid particles.
  - No shearing stress (no velocity gradients)
  - Only normal stress exists (pressure)

Pressure At a Point

- Pressure is an infinitesimal normal compressive force divided by the infinitesimal area over which it acts.
- From Newton's Second Law (for x- and y-directions):



Pressure at a point in a fluid.

$$\Sigma F_x = ma_x: \quad p_x\Delta y - p\Delta s \sin\theta = \rho\frac{\Delta x\Delta y}{2}a_x$$

$$\Sigma F_y = ma_y: \quad p_y\Delta x - \rho g\frac{\Delta x\Delta y}{2} - p\Delta s\cos\theta = \rho\frac{\Delta x\Delta y}{2}a_y$$

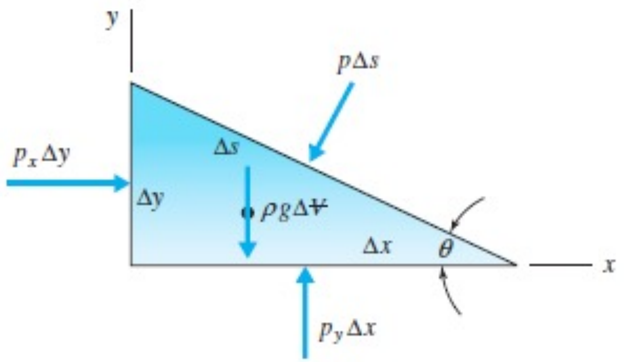
$$p_x - p = \frac{\rho a_x}{2}\Delta x$$

$$p_y - p = \frac{\rho(a_y + g)}{2}\Delta y$$

As the element goes to a point (Δx, $\Delta y \rightarrow 0$)

$$\boxed{p_x = p_y = p}$$

Pressure At a Point

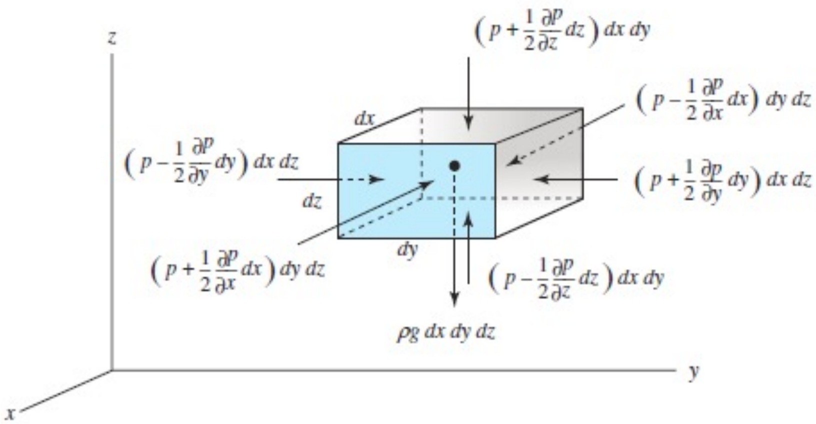


Pressure at a point in a fluid.

$$p_x = p_y = p$$

- Pressure in a fluid is **constant** at a point.
- Pressure is a **scalar** function.
- It acts **equally in all directions** at a point for both static and dynamic fluids.

Pressure Variation



Forces acting on an infinitesimal element that is at rest in the *xyz*-reference frame. The reference frame may be accelerating or rotating.

- Using Newton's Second Law, pressures at each of the sides:

$$dp = \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy + \frac{\partial p}{\partial z}dz$$

- The pressure at a face that is $\frac{1}{2}dx$ from the center is:

$$p\left(x + \frac{dx}{2}, y, z\right) = p(x, y, z) + \frac{\partial p}{\partial x}\frac{dx}{2}$$

- The pressure differential in any direction would be:

$$dp = -\rho a_x dx - \rho a_y dy - \rho(a_z + g)dz$$

Pressure

- The pressure differential (from the previous slide) is:

$$dp = -\rho a_x dx - \rho a_y dy - \rho(a_z + g)dz$$

- At rest, there is no acceleration (a = 0):

$$dp = -\rho g\, dz$$

$$\frac{dp}{dz} = -\gamma$$

No pressure variation in the x- and y-directions (horizontal plane). Pressure varies in the z-direction only (dp is negative if dz is positive).

Pressure decreases as we move up and increases as we move down.

Pressure

Pressure in Liquids at Rest

- At a distance h below a free surface, the pressure is:

$$p = \gamma h$$

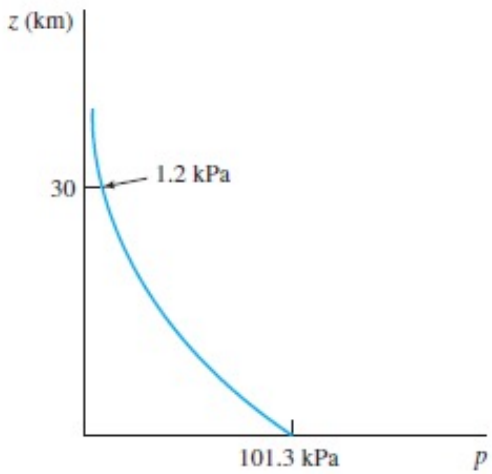
p = 0 at h = 0.

Pressure

Pressure in the Atmosphere

$$p = p_{\text{atm}}\left(\frac{T_0 - \alpha z}{T_0}\right)^{g/\alpha R}$$

- The equation above shows that pressure varies with elevation (z).



Pressure

The atmospheric pressure is given as 680 mm Hg at a mountain location. Convert this to kilopascals and meters of water. Also, calculate the pressure decrease due to 500-m elevation increase, starting at 2000 m elevation, assuming constant density.

Solution

Use Eq. 2.4.4 and find, using $S_{Hg} = 13.6$ with Eq. 1.5.2,

$$\begin{aligned} p &= \gamma_{Hg} h \\ &= (9.81 \text{ kN/m}^3 \times 13.6) \times 0.680 \text{ m} = \underline{90.7 \text{ kPa}} \end{aligned}$$

To convert this to meters of water, we have

$$\begin{aligned} h &= \frac{p}{\gamma_{H_2O}} \\ &= \frac{90.7}{9.810} = \underline{9.25 \text{ m of water}} \end{aligned}$$

To find the pressure decrease, we use Eq. 2.4.3 and find the density in Table B.3:

$$\begin{aligned} \Delta p &= -\gamma \Delta z = -\rho g \Delta z \\ &= -1.007 \text{ kg/m}^3 \times 9.81 \text{ m/s}^2 \times 500 \text{ m} = \underline{-4940 \text{ Pa}} \end{aligned}$$

where we used $\text{kg} = \text{N} \cdot \text{s}^2/\text{m}$.

Note: Since gravity is known to three significant digits, we express the answer to three significant digits.

Pressure

Assume an isothermal atmosphere and approximate the pressure at 10 000 m. Calculate the percent error when compared with the values using Eq. 2.4.8 and from Appendix B.3. Use a temperature of 256 K, the temperature at 5000 m.

Solution

Integrate Eq. 2.4.5 assuming that T is constant, as follows:

$$\int_{101}^{p} \frac{dp}{p} = -\frac{g}{RT}\int_{0}^{z} dz$$

$$\ln\frac{p}{101} = -\frac{gz}{RT} \qquad \text{or} \qquad p = 101e^{-gz/RT}$$

Substituting $z = 10\,000$ m and T = 256 K, there results

$$p = 101e^{-9.81 \times 10\,000/(287 \times 256)}$$
$$= 26.57 \text{ kPa}$$

Using Eq. 2.4.8 we have

$$p = p_{\text{atm}}\left(\frac{T_0 - \alpha z}{T_0}\right)^{g/\alpha R}$$
$$= 101\left(\frac{288 - 0.0065 \times 10\,000}{288}\right)^{9.81/0.0065 \times 287} = 26.3 \text{ kPa}$$

The actual pressure at 10 000 m is found from Table B.3 to be 26.50 kPa. Hence the percent errors are

$$\%\,\text{error} = \left(\frac{26.57 - 26.3}{26.3}\right) \times 100 = \underline{1.03\%}$$

$$\%\,\text{error} = \left(\frac{26.57 - 26.50}{26.50}\right) \times 100 = \underline{0.26\%}$$

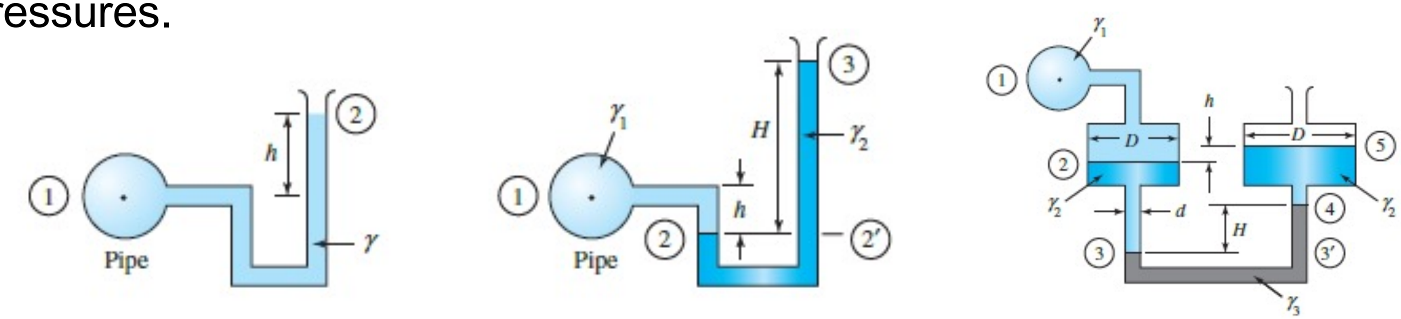
Because the error is so small, we often assume the atmosphere to be isothermal. *Note:* When evaluating gz/RT we use $R = 287$ J/kg·K, not 0.287 kJ/kg·K. To observe that gz/R is dimensionless, which it must be since it is an exponent, use N = kg·m/s² so that

$$\left[\frac{gz}{RT}\right] = \frac{(\text{m/s}^2)\text{m}}{(\text{J/kg}\cdot\text{K})\text{K}} = \frac{\text{m}^2/\text{s}^2}{\text{N}\cdot\text{m/kg}} = \frac{\text{m}^2/\text{s}^2}{(\text{kg}\cdot\text{m}^2/\text{s}^2)/\text{kg}} = \frac{\text{m}^2/\text{s}}{\text{m}^2/\text{s}}$$

Manometer

Manometers

- Manometers are instruments that use columns of liquid to measure pressures.

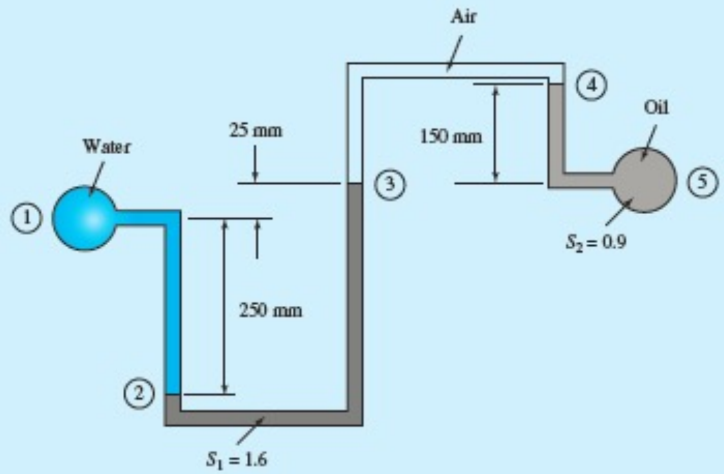


Manometers: (a) U-tube manometer (small pressures); (b) U-tube manometer (large pressures); (c) micromanometer (very small pressure changes).

- Center: Large pressures can be measured using a liquid with large γ_2.
- Right: Very small pressures can be measured as small pressure changes in p_1, leading to a relatively large deflection H.

Manometer

Water and oil flow in horizontal pipelines. A double U-tube manometer is connected between the pipelines, as shown in Figure E2.3. Calculate the pressure difference between the water pipe and the oil pipe.



Solution

We first identify the relevant points as shown in the figure. Begin at point ① and add pressure when the elevation decreases and subtract pressure when the elevation increases until point ⑤ is reached. Using $\gamma = \gamma_{water}$, we have.

$$p_1 + \gamma(z_1 - z_2) - \gamma S_1(z_3 - z_2) - \gamma S_{air}(z_4 - z_3) + \gamma S_2(z_4 - z_5) = p_5$$

where $\gamma = 9810\ \mathrm{N/m^3}$, $S_1 = 1.6$, $S_2 = 0.9$, and $S_{air} \approx 0$. Thus

$$p_1 - p_5 = 9810\left(-\frac{250}{1000} + 1.6 \times \frac{275}{1000} + 0 \times \frac{150}{1000} - 0.9 \times \frac{150}{1000}\right)$$
$$= 542\ \mathrm{Pa}$$

Note that by neglecting the weight of the air, the pressure at point 3 is equal to the pressure at point 4.

Manometer

For a given condition the liquid levels in Figure 2.7c are $z_1 = 0.95\,\text{m}$, $z_2 = 0.70$ m, $z_3 = 0.52$ m, $z_4 = 0.65$ m, and $z_5 = 0.72$ m. Further, $\gamma_1 = 9810\,\text{N/m}^3$, $\gamma_2 = 11500\,\text{N/m}^3$, and $\gamma_3 = 14\,000\,\text{N/m}^3$. The diameters are $D = 0.2$ m and $d = 0.01$ m. (a) Calculate the pressure p_1 in the pipe, (b) the change in H if p_1 increases by 100 Pa, and (c) the change in h of the manometer of Figure 2.7a if $h = 0.5$ m of water and $\Delta p_1 = 100$ Pa.

Solution

(a) Referring to Figure 2.7c, we have

$$h = 0.72 - 0.70 = 0.02\text{ m}$$
$$H = 0.6 - 0.52 = 0.13\text{ m}$$

Substituting the given values into Eq. (2.4.16) leads to

$$\begin{aligned} p_1 &= \gamma_1(z_2 - z_1) + \gamma_2 h + (\gamma_3 - \gamma_2)H \\ &= 9810(0.70 - 0.95) + 11500(0.02) + (14\,000 - 11500)(0.13) \\ &= \underline{-1898\text{ Pa}} \end{aligned}$$

(b) If the pressure p_1 is increased by 100 Pa to $p_1 = -1798$ Pa, the change in H is, using Eq. 2.4.19,

$$\Delta H = \Delta p_1 \frac{2D^2/d^2}{-\gamma_1 + 2\gamma_2 + 2(\gamma_3 - \gamma_2)D^2/d^2}$$

$$\Delta H = 100 \frac{2(20^2)}{-9810 + 2(11\,500) + 2(14\,000 - 11\,500) \times 20^2} = \underline{0.0397\text{ m}}$$

Thus H increases by 3.97 cm as a result of increasing the pressure by 100 Pa.

(c) For the manometer in Figure 2.7a, the pressure p_1 is given by $p = \gamma h$. Assume that initially $h = 0.50$ m. Thus the pressure initially is

$$p_1 = 9810 \times 0.50 = 4905\text{ Pa}$$

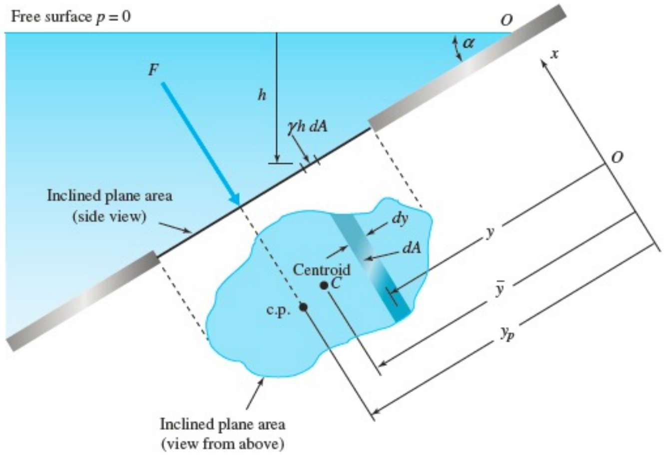
Now if p_1 is increased by 100 Pa to 5005 Pa, h can be found:

$$p_1 = \gamma h$$
$$h = \frac{p_1}{\gamma} = \frac{5005}{9810} = 0.510\text{ m.} \qquad \therefore \Delta h = 0.510 - 0.5 = \underline{0.01\text{ m}}$$

Thus an increase of 100 Pa increases h by 1 cm in the manometer shown in part (a), 25% of the change in the micromanometer.

Hydrostatic force on plane surface

Forces on Plane Areas



Force on an inclined plane area.

- The total force of a liquid on a plane surface is:

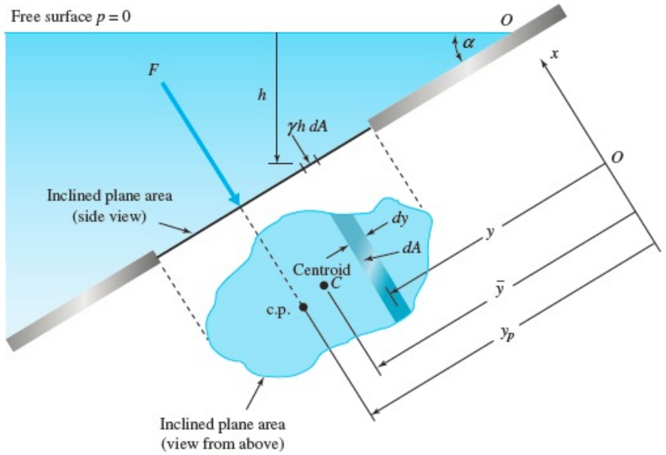
$$F = \int_A p\, dA$$

- After knowing the equation for pressure (P = γh):

$$F = \int_A \gamma h\, dA$$
$$= \gamma \sin\alpha \int_A y\, dA$$

Hydrostatic force on plane surface

Forces on Plane Areas



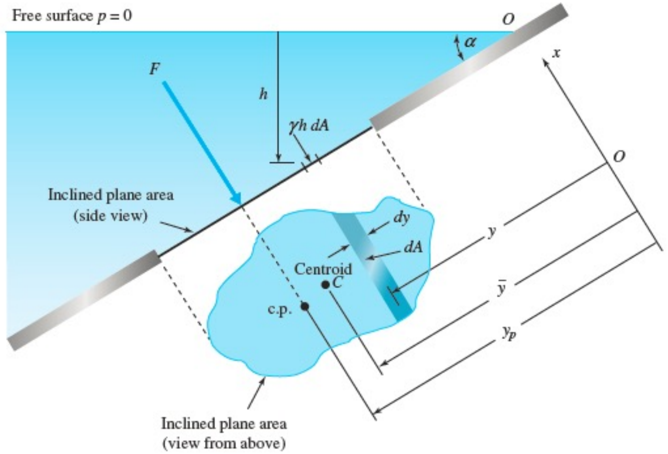
Force on an inclined plane area.

$$F = \gamma \bar{y} A \sin\alpha$$
$$= \gamma \bar{h} A = p_C A$$

$\bar{h}$: Vertical distance from the free surface to the centroid of the area

p_C: Pressure at the centroid

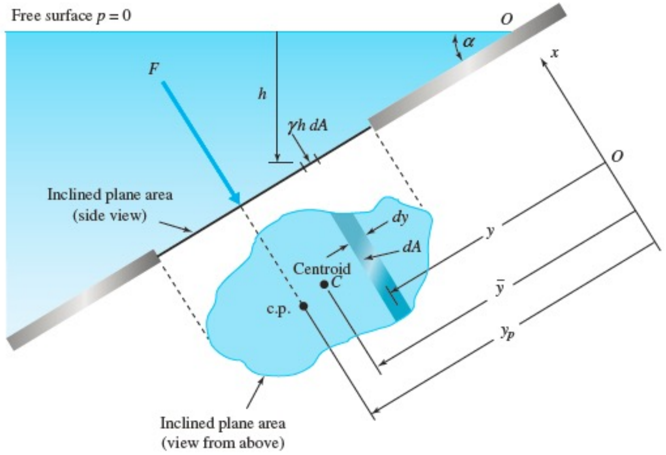
Hydrostatic force on plane surface



Force on an inclined plane area.

- The center of pressure is the point where the resultant force acts:
    - Sum of moments of all infinitesimal pressure forces on an area, A, equals the moment of the resultant force.

Hydrostatic force on plane surface



Force on an inclined plane area.

$$x_p = \bar{x} + \frac{\bar{I}_{xy}}{A\bar{y}}$$

$$y_p = \frac{\gamma(\bar{I} + A\bar{y}^2)\sin\alpha}{\gamma\bar{y}A\ \sin\alpha} = \bar{y} + \frac{\bar{I}}{A\bar{y}}$$

$\bar{y}$: Measured parallel to the plane area to the free surface

- The moments of area can be found using:

$$I_x = \int_A y^2\, dA \qquad I_{xy} = \int_A xy\, dA$$

$$I_x = \bar{I} + A\bar{y}^2 \qquad I_{xy} = \bar{I}_{xy} + A\bar{x}\,\bar{y}$$

Hydrostatic force on plane surface

A plane area of 800 cm × 800 cm acts as an escape hatch on a submersible in the Great Lakes. If it is on a 45° angle with the horizontal, what force applied normal to the hatch at the bottom edge is needed to just open the hatch, if it is hinged at the top edge when the top edge is 10 m below the surface? The pressure inside the submersible is assumed to be atmospheric.

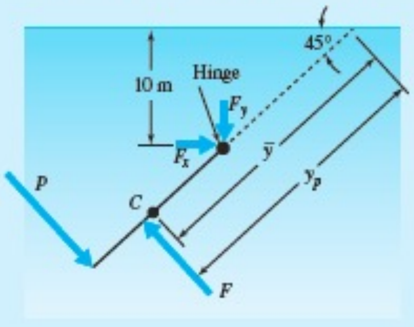


Figure E2.5

Solution

First, a sketch of the hatch would be very helpful, as in Figure E2.5. The force of the water acting on the hatch is

$$F = \gamma \bar{h} A$$
$$= 9810(10 + 0.4 \times \sin 45°)(0.8 \times 0.8) = 64\,560 \text{ N}$$

The distance $\bar{y}$ is

$$\bar{y} = \frac{\bar{h}}{\sin 45°} = \frac{10 + 0.4 \times \sin 45°}{\sin 45°} = 14.542 \text{ m}$$

so that

$$y_p = \bar{y} + \frac{\bar{I}}{A\bar{y}}$$
$$= 14.542 + \frac{0.8 \times 0.8^3/12}{(0.8 \times 0.8) \times 14.542} = 14.546 \text{ m}$$

Taking moments about the hinge provides the needed force P to open the hatch:

$$0.8\,P = \left(y_p - \bar{y} + 0.4\right) F$$
$$\therefore\ P = \frac{14.546 - 14.542 + 0.4}{0.8} 64\,560 = \underline{32\,610 \text{ N}}$$

Alternatively, we could have sketched the pressure prism, composed of a rectangular volume and a triangular volume. Moments about the top hinge would provide the desired force.

Hydrostatic force on plane surface

Find the location of the resultant force F of the water on the triangular gate and the force P necessary to hold the gate in the position shown in Figure E2.6a. Neglect the weight of the gate, as usual.

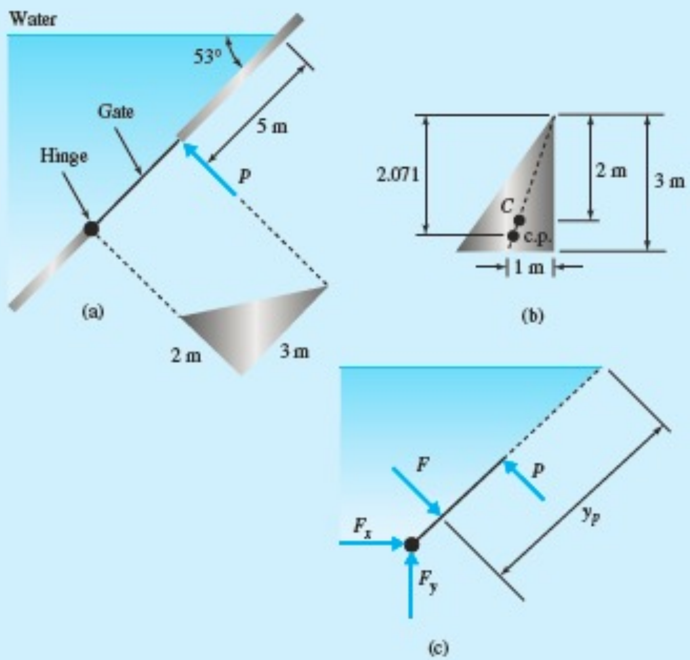


Figure E2.6

Solution

First we draw a free-body diagram of the gate, including all the forces acting on the gate (Figure E2.6c). The centroid of the gate is shown in Figure E2.6b. The y-coordinate of the location of the resultant F can be found using Eq. 2.4.28 as follows:

$$\bar{y} = 2 + 5 = 7$$

$$y_p = \bar{y} + \frac{\bar{I}}{A\bar{y}}$$

$$= 7 + \frac{2 \times 3^3/36}{3 \times 7} = \underline{7.071 \text{ m}}$$

To find x_p we could use Eq. 2.4.32. Rather than that, we recognize that the resultant force must act on a line connecting the vertex and the midpoint of the opposite side since each infinitesimal force acts on this line (the moment of the resultant must equal the moment of its components). Thus, using similar triangles we have

$$\frac{x_p}{1} = \frac{2.071}{3}$$

$$\therefore\ x_p = \underline{0.690 \text{ m}}$$

The coordinates x_p and y_p locate where the force due to the water acts on the gate.

If we take moments about the hinge, assumed to be frictionless, we can determine the force P necessary to hold the gate in the position shown:

$$\Sigma M_{\text{hinge}} = 0$$

$$\therefore\ 3 \times P = (3 - 2.071)F$$

$$= 0.929 \times \gamma \bar{h} A$$

$$= 0.929 \times 9810 \times (7 \sin 53°) \times 3$$

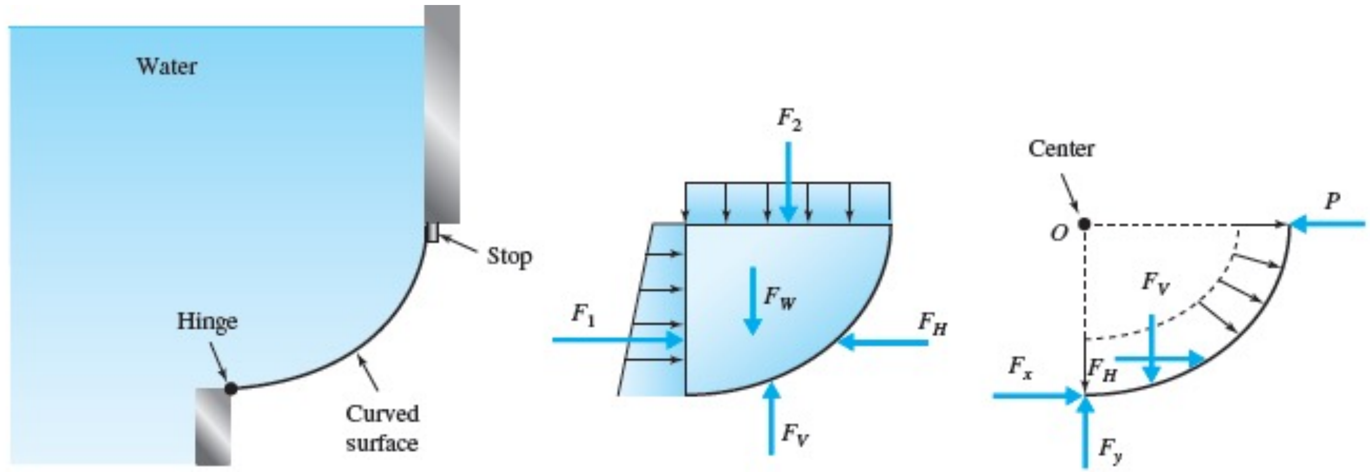
where $\bar{h}$ is the vertical distance from the centroid to the free surface. Hence

$$P = 50\,900 \text{ N} \quad \text{or} \quad \underline{50.9 \text{ kN}}$$

Hydrostatic force on curve surface

Forces on Curved Surfaces

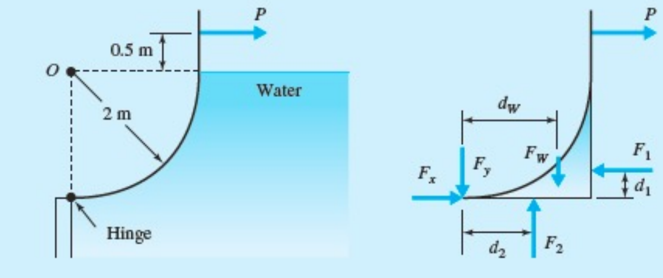
- Direct integration cannot find the force due to the hydrostatic pressure on a curved surface.
- A free-body diagram containing the curved surface and surrounding liquid needs to be identified.



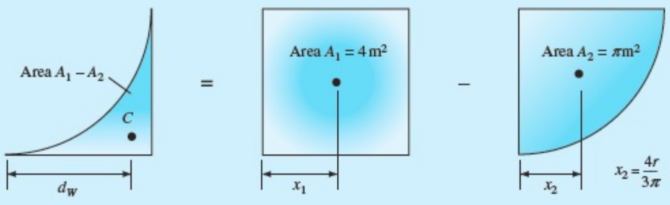
Forces acting on a curved surface: (a) curved surface; (b) free-body diagram of water and gate; (c) free-body diagram of gate only.

Hydrostatic force on curve surface

Calculate the force P necessary to hold the 4-m-wide gate in the position shown in Figure E2.7a. Neglect the weight of the gate.



(a) (b)



(c) (d) (e)

Figure E2.7

Solution

The first step is to draw a free-body diagram. One choice is to select the gate and the water directly below the gate, as shown in Figure E2.7b. To calculate P, we must determine F_1, F_2, F_W, d_1, d_2, and d_W; then moments about the hinge will allow us to find P. The force components are given by

$$\begin{aligned} F_1 &= \gamma \bar{h}_1 A_1 \\ &= 9810 \times 1 \times (2 \times 4) = 78\,480\ \text{N} \\ F_2 &= \gamma \bar{h}_2 A_2 \\ &= 9810 \times 2 \times (2 \times 4) = 156\,960\ \text{N} \\ F_W &= \gamma \mathcal{V}_{\text{water}} \\ &= 9810 \times 4\left(4 - \frac{\pi \times 2^2}{4}\right) = 33\,700\ \text{N} \end{aligned}$$

The distance d_W is the distance to the centroid of the volume. It can be determined by considering the area as the difference of a square and a quarter circle as shown in Figure E2.7c, d, and e. Moments of areas yield

$$\begin{aligned} d_W(A_1 - A_2) &= x_1 A_1 - x_2 A_2 \\ d_W &= \frac{x_1 A_1 - x_2 A_2}{A_1 - A_2} \\ &= \frac{1 \times 4 - (4 \times 2/3\pi) \times \pi}{4 - \pi} = 1.553\ \text{m} \end{aligned}$$

The distance $d_2 = 1\,\text{m}$. Because F_1 is due to a triangular pressure distribution (see Figure 2.9), d_1 is given by

$$d_1 = \frac{1}{3}(2) = 0.667\ \text{m}$$

Summing moments about the frictionless hinge gives

$$\begin{aligned} 2.5P &= d_1 F_1 + d_2 F_2 - d_W F_W \\ P &= \frac{0.667 \times 78.5 + 1 \times 157.0 - 1.553 \times 33.7}{2.5} = \underline{62.8\ \text{kN}} \end{aligned}$$

Hydrostatic force on curve surface

Rather than the somewhat tedious procedure above, we could observe that all the infinitesimal forces that make up the resultant force $(\mathbf{F}_H + \mathbf{F}_V)$ acting on the circular arc pass through the center O, as noted in Figure 2.11c. Since each infinitesimal force passes through the center, the resultant force must also pass through the center. Hence we could have located the resultant force $(\mathbf{F}_H + \mathbf{F}_V)$ at point O. If F_V and F_H were located at O, F_V would pass through the hinge, producing no moment about the hinge. Then, realizing that $F_H = F_1$ and summing moments about the hinge gives

$$2.5P = 2F_H$$

Therefore,

$$P = 2 \times \frac{78.48}{2.5} = \underline{\underline{62.8 \text{ kN}}}$$

This was obviously much simpler. All we needed to do was calculate F_H and then sum moments!

Hydrostatic force on curve surface

Find the force P needed to hold the gate in the position shown in Figure E2.8a if P acts 3 m from the y-axis. The parabolic gate is 150 cm wide.

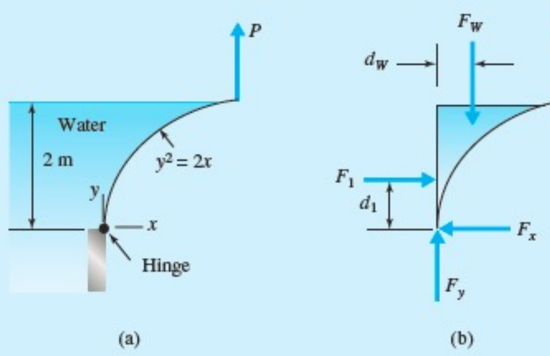


Figure E2.8

Solution

A free-body diagram of the gate and the water directly above the gate is shown in Figure E2.8b. The forces are found to be

$$F_1 = \gamma \bar{h} A = 9810 \times 1 \times (2 \times 1.5) = 29\,430\text{ N}$$

$$F_W = \gamma V = 9810 \int_0^2 1.5x\,dy = 14\,715 \int_0^2 \frac{y^2}{2} dy = 14\,715 \frac{2^3}{6} = 19\,620\text{ N}$$

The distance d_1 is $\frac{1}{3}(2) = 0.667$ m since the top edge is in the free surface. The distance d_W through the centroid is found using a horizontal strip:

$$d_W = \frac{\int_0^2 x(x/2)dy}{\int_0^2 x\,dy} = \frac{\frac{1}{8}\int_0^2 y^4\,dy}{\frac{1}{2}\int_0^2 y^2\,dy} = \frac{1}{4}\frac{2^5/5}{2^3/3} = 0.6\text{ m}$$

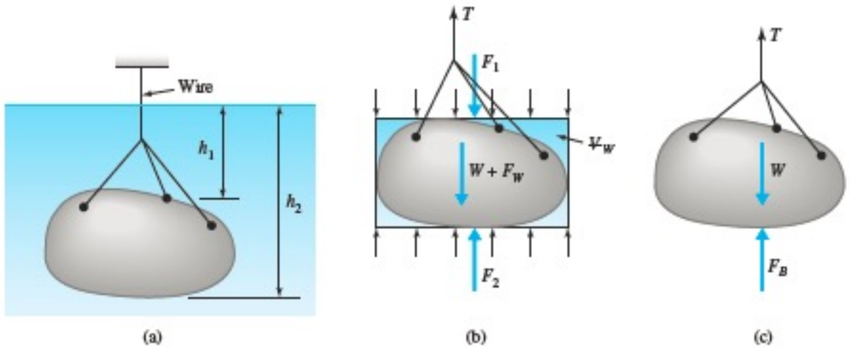
Sum moments about the hinge and find P as follows:

$$3P = d_1 F_1 + d_W F_W = 0.667 \times 29\,430 + 0.6 \times 19\,620 \qquad \therefore P = \underline{10\,470\text{ N}}$$

Bouyancy

Buoyancy (Archimedes' principle)

- **Buoyancy force on an object equals the weight of displaced liquid.**



Forces on a submerged body: (a) submerged body; (b) free-body diagram; (c) free body showing the buoyant force F_B.

$$F_B = \gamma \forall_{\text{displaced liquid}}$$

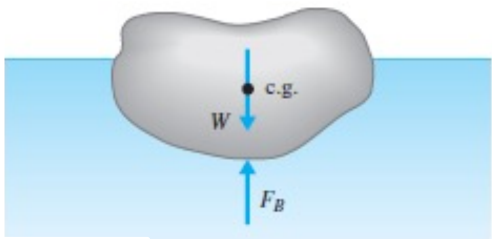
V is the volume of displaced fluid and W is the weight of the floating object.

$$F_B = W$$

Bouyancy

Buoyancy (Archimedes' principle)

$$W = \gamma_{water} V$$



Forces on a floating object.

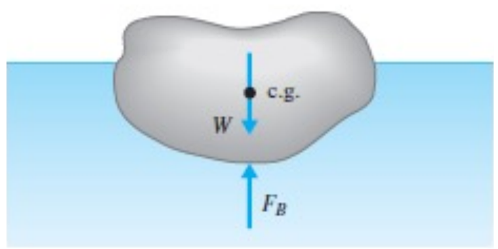
- The buoyant force acts through the centroid of the displaced liquid volume.

- An application of this would be a hydrometer that is used to measure the specific gravity of liquids.
    - For pure water, this is 1.0

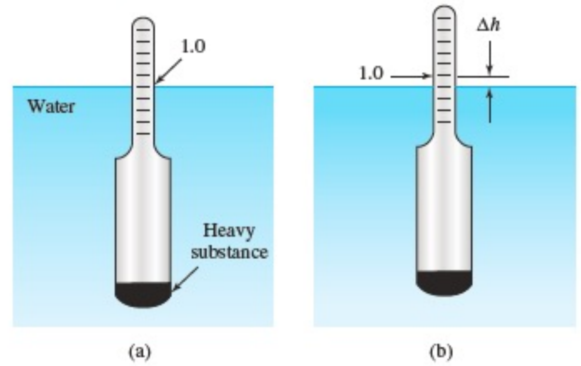
Bouyancy

Buoyancy (Hydrometers)

$$W = \gamma_{\text{water}} \rlap{-}V$$



Forces on a floating object.



Hydrometer: (a) in water; (b) in an unknown liquid.

$$\Delta h = \frac{\rlap{-}V}{A}\left(1 - \frac{1}{S_x}\right)$$

- The displaced height, h, can be found as shown in the above equation.
  - A: Cross-sectional area of the stem
  - $S_x = \dfrac{\gamma_x}{\gamma_{water}}$
  - For a given hydrometer, $\rlap{-}V$ and A are fixed.

Bouyancy

The specific weight and the specific gravity of a body of unknown composition are desired. Its weight in air is found to be 890 N, and in water it weighs 667 N.

Solution

The volume is found from a force balance when submerged as follows (see Figure 2.12c):

$$T = W - F_B$$

$$667 = 890 - 9810\forall \qquad \therefore \forall = 0.02273 \text{ m}^3$$

The specific weight is then

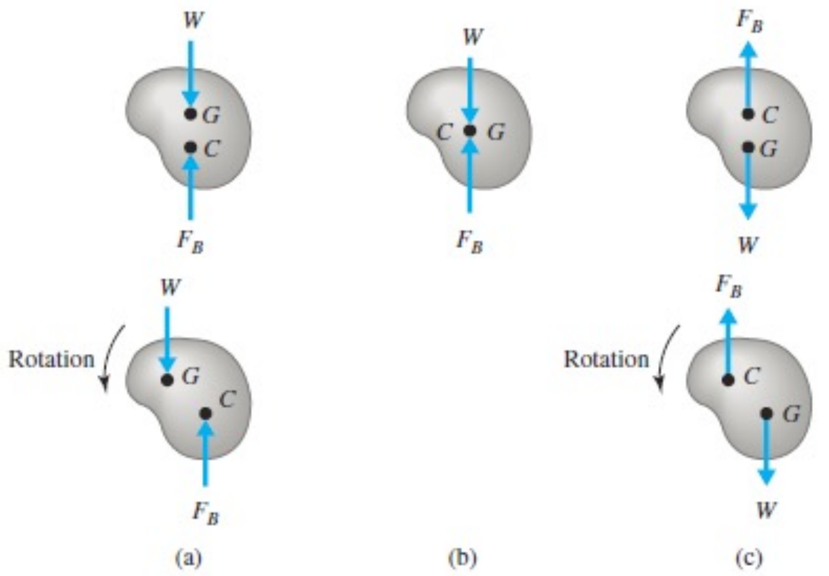
$$\gamma = \frac{W}{\forall} = \frac{890}{0.02273} = \underline{39\,155 \text{ N/m}^3}$$

The specific gravity is found to be

$$S = \frac{\gamma}{\gamma_{\text{water}}} = \frac{39\,155}{9810} = \underline{3.99}$$

Stability

Stability

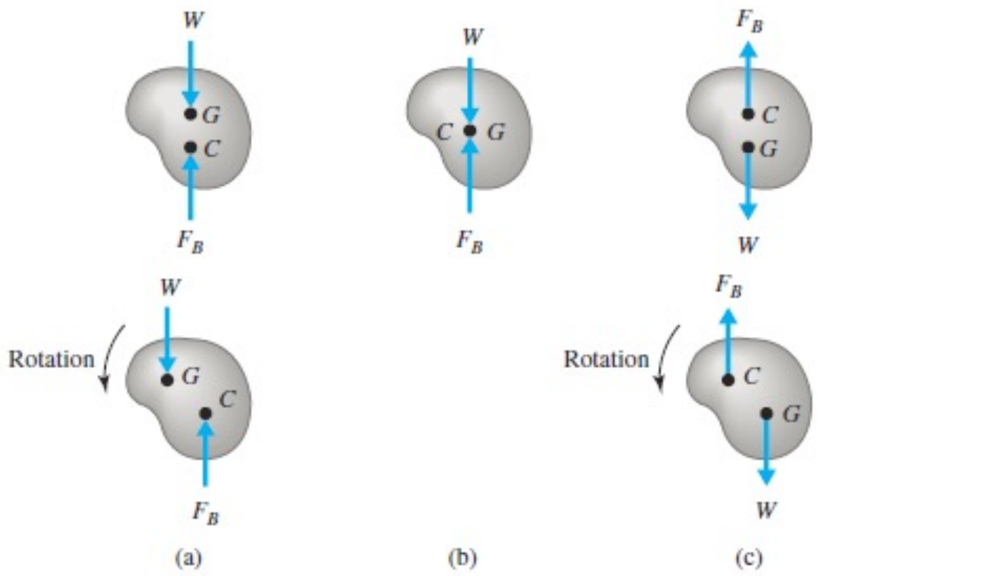


Stability of a submerged body: (a) unstable; (b) neutral; (c) stable.

- In (a) the center of gravity of the body is above the centroid C (center of buoyancy), so a small angular rotation leads to a moment that increases rotation: unstable.
- (b) shows neutral stability as the center of gravity and the centroid coincide.
- In (c), as the center of gravity is below the centroid, a small angular rotation provides a restoring moment and the body is stable.

Stability

Stability



Stability of a submerged body: (a) unstable; (b) neutral; (c) stable.

- The **metacentric height $\overline{GM}$** is the distance from G to the point of intersection of the buoyant force before rotation with the buoyant force after rotation.
- If $\overline{GM}$ is positive: Stable
- If $\overline{GM}$ is negative: Unstable

Stability

A 0.25-m-diameter cylinder is 0.25 m long and composed of material with specific weight 8000 N/m^3. Will it float in water with the ends horizontal?

Solution

With the ends horizontal, I_o will be the second moment of the circular cross section,

$$I_o = \frac{\pi d^4}{64} = \frac{\pi \times 0.25^4}{64} = 0.000192\ \text{m}^4$$

The displaced volume will be

$$\forall = \frac{W}{\gamma_{\text{water}}} = \frac{8000 \times \pi/4 \times 0.25^2 \times 0.25}{9810} = 0.0100\ \text{m}^3$$

The depth the cylinder sinks in the water is

$$depth = \frac{\forall}{A} = \frac{0.01}{\pi \times 0.25^2/4} = 0.204\ \text{m}$$

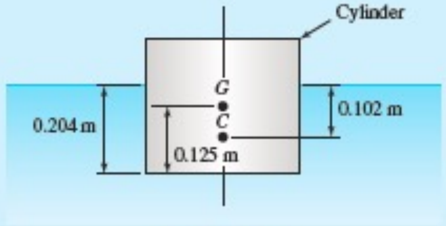


Figure E2.10

Hence, the distance $\overline{CG}$, as shown in Figure E2.10, is

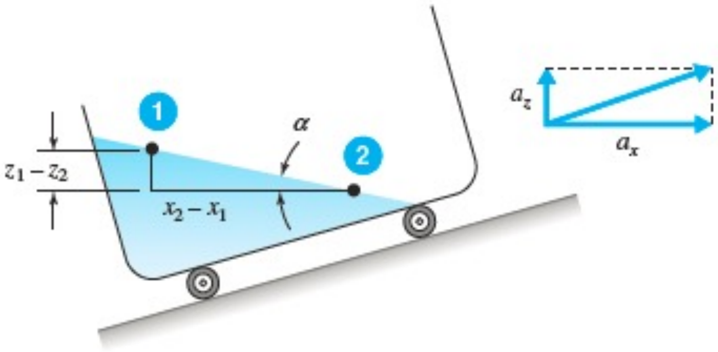
$$\overline{CG} = 0.125 - \frac{0.204}{2} = 0.023\ \text{m}$$

Finally,

$$\overline{GM} = \frac{0.000192}{0.01} - 0.023 = -0.004\ \text{m}$$

This is a negative value showing that *the cylinder will not float with ends horizontal*. It would undoubtedly float on its side.

Linearly Accelerating Containers



Linearly accelerating tank.

- The pressure differential equation

$$dp = -\rho a_x dx - \rho a_y dy - \rho(a_z + g)dz$$

- When the fluid is at rest relative to a reference frame that is linearly accelerating with horizontal (a_x) and vertical (a_z) components:

$$dp = -\rho a_x dx - \rho(g + a_z)dz$$

- As points 1 and 2 lie on a constant-pressure line:

$$\frac{z_1 - z_2}{x_2 - x_1} = \tan\alpha = \frac{a_x}{g + a_z}$$

Where α is the angle that the constant-pressure line makes with the horizontal.

Linearly Accelerating Containers

The tank shown in Figure E2.11a is accelerated to the right. Calculate the acceleration a_x needed to cause the free surface, shown in Figure E2.11b, to touch point A. Also, find p_B and the total force acting on the bottom of the tank if the tank width is 1 m.

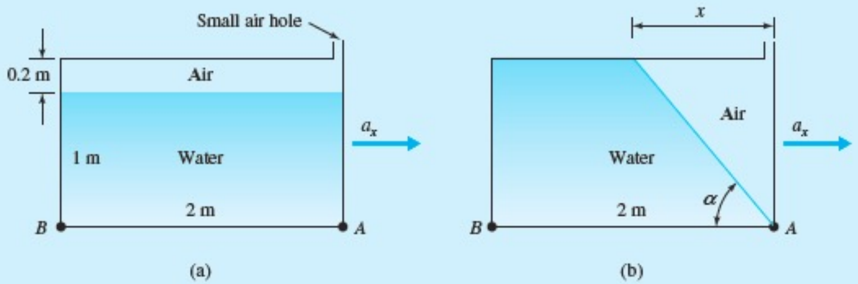


Figure E2.11

Solution

The angle the free surface takes is found by equating the air volume (actually, areas since the width is constant) before and after since no water spills out:

$$0.2 \times 2 = \frac{1}{2}(1.2x)$$
$$x = 0.667\ \text{m}$$

The quantity $\tan\alpha$ can now be found. It is

$$\tan\alpha = \frac{1.2}{0.667} = 1.8$$

Using Eq. 2.5.3, we find a_x to be, letting $a_z = 0$,

$$a_x = g\tan\alpha$$
$$= 9.81 \times 1.8 = \underline{17.66\ \text{m/s}^2}$$

We can find the pressure at B by noting the pressure dependence on x. At A, the pressure is zero. Hence, Eq. 2.5.2 yields

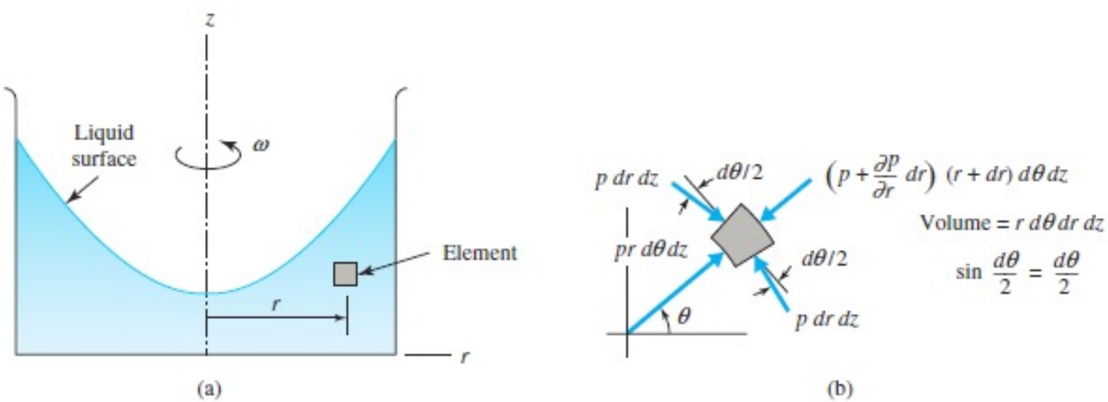
$$p_B - \cancelto{0}{p_A} = -\rho a_x(x_B - x_A)$$
$$p_B = -1000 \times 17.66(-2)$$
$$= 35\,300\ \text{Pa} \quad \text{or} \quad \underline{35.3\ \text{kPa}}$$

To find the total force acting on the bottom of the tank, we realize that the pressure distribution is decreasing linearly from $p = 35.3\,\text{kPa}$ at B to $p = 0\,\text{kPa}$ at A. Hence, we can use the average pressure over the bottom of the tank:

$$F = \frac{p_B + p_A}{2} \times \text{area}$$
$$= \frac{35\,300 + 0}{2} \times 2 \times 1 = \underline{35\,300\ \text{N}}$$

Rotating Containers

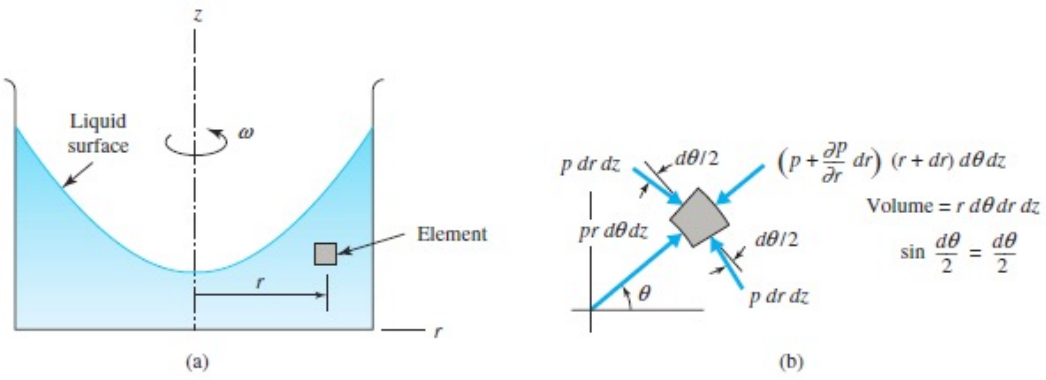
- For a liquid in a rotating container (cross-section shown):



Rotating container: (a) liquid cross section; (b) top view of element.

- In a short time, the liquid reaches static equilibrium with respect to the container and the rotating rz-reference frame.
- Horizontal rotation will not affect the pressure distribution in the vertical direction.
- No variation in pressure with respect to the θ-coordinate.

Rotating Containers



Rotating container: (a) liquid cross section; (b) top view of element.

- Between two points (r_1,z_1) and (r_2,z_2) on a rotating container, the static pressure variation is:

$$p_2-p_1=\frac{\rho\omega^2}{2}\left(r_2^2-r_1^2\right)-\rho g\left(z_2-z_1\right)$$

- If two points are on a constant-pressure surface (e.g., free surface) with point 1 on the z-axis $[r_1 = 0]$:

$$\frac{\omega^2 r_2^2}{2}=g\left(z_2-z_1\right)$$

- The free surface is a **paraboloid of revolution**.

Rotating Containers

The cylinder shown in Figure E2.12 is rotated about its centerline. Calculate the rotational speed that is necessary for the water to just touch the origin O. Also, find the pressures at A and B.

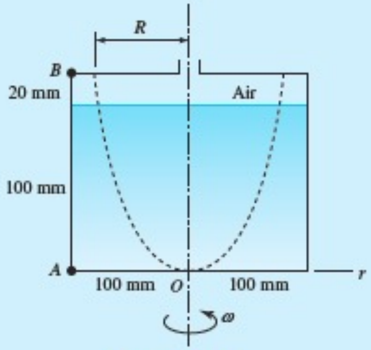


Figure E2.12

Solution

Since no water spills from the container, the air volume remains constant, that is,

$$\pi r^2 h = (\pi R^2) H/2$$

$$\pi \times 0.1^2 \times 0.02 = \frac{1}{2}\pi R^2 \times 0.12$$

where we have used the fact that the volume of a paraboloid of revolution is one-half that of a circular cylinder with the same height and radius. This gives the value

$$R = 57.7 \text{ mm}$$

Using Eq. 2.6.5 with $r_2 = R$, we have

$$\frac{\omega^2 \times 0.0577^2}{2} = 9.81 \times 0.12$$

$$\therefore \ \omega = \underline{26.6 \text{ rad/s}}$$

To find the pressure at point A, we simply calculate the pressure difference between A and O. Using Eq. 2.6.4 with $r_2 = r_A = 0.1\text{m}$, $r_1 = r_0 = 0$, and $p_1 = p_0 = 0$, there results

$$p_A = \frac{\rho\omega^2}{2}(r_A^2 - r_0^2) = \frac{1000 \text{ kg/m}^3 \times (26.6 \text{ rad/s})^2}{2} \times 0.1^2 \text{m}^2 = 3540 \text{ Pa} \quad \text{or} \quad \underline{3.54 \text{ kPa}}$$

using $\text{kg} = \text{N·s}^2/\text{m}$. The pressure at B can be found by applying Eq. 2.6.4 to points A and B. This equation simplifies to

$$p_B - p_A = -\rho g(z_B - z_A)$$

Hence

$$p_B = 3540 - 1000 \text{ kg/m}^3 \times 9.81 \text{ m/s}^2 \times 0.12 \text{ m} = 2360 \text{ Pa} \quad \text{or} \quad \underline{2.36 \text{ kPa}}$$

Summary

- Pressure variation in the vertical (z-direction) in a constant density fluid is:

$$\Delta p = -\gamma \Delta z$$

- The force on a plane would be:

$$F = \gamma \bar{h} A$$

Where $\bar{h}$ is the vertical distance to the centroid of the area

- The force is located a distance from the free-surface to the center of pressure:

$$y_p = \bar{y} + \frac{\bar{I}}{A\bar{y}}$$

Where $\bar{I}$ is the centroidal axis

Summary

- In a container rotating with angular velocity, ω, a constant-pressure surface is:

$$\frac{1}{2}\omega^2 r_2^2 = g(z_2 - z_1)$$

Point 1 is on the axis of rotation and point 2 is on the constant-pressure surface