Fluids in Motion

Introduction

- General equations of motion in fluid flow are very difficult to solve.
  - Need simplifying assumptions.
  - In some cases viscosity can affect flow significantly, whereas in others it can be neglected.

Description of Fluid Motion

Lagrangian and Eulerian Descriptions of Motion

- When describing flow fields, think of individual particles.
  - Each of these is considered to be a small mass of fluid with a large number of molecules (occupying a small volume).

- Incompressible Fluid: Volume doesn't change in magnitude, but fluid may deform.
- Compressible Fluid: As the volume deforms, the magnitude changes.

- **Lagrangian Description**: Description of fluid motion (position, velocity, acceleration), where individual particles are observed as a function of time.
  - (x_0,y_0,z_0,t), etc.
  - Becomes difficult as the number of particles becomes very large in simple fluid flows.

Description of Fluid Motion

Lagrangian and Eulerian Descriptions of Motion

- **Eulerian Description**: Description of fluid motion where the flow properties are functions of both space and time.
  - In Cartesian coordinates, velocity: $V = V(x, y, z, t)$

Description of Fluid Motion

Lagrangian and Eulerian Descriptions of Motion

- **STEADY FLOW**: Flow quantities do not depend on time.

$$\frac{\partial \mathbf{V}}{\partial t} = 0 \qquad \frac{\partial p}{\partial t} = 0 \qquad \frac{\partial \rho}{\partial t} = 0$$

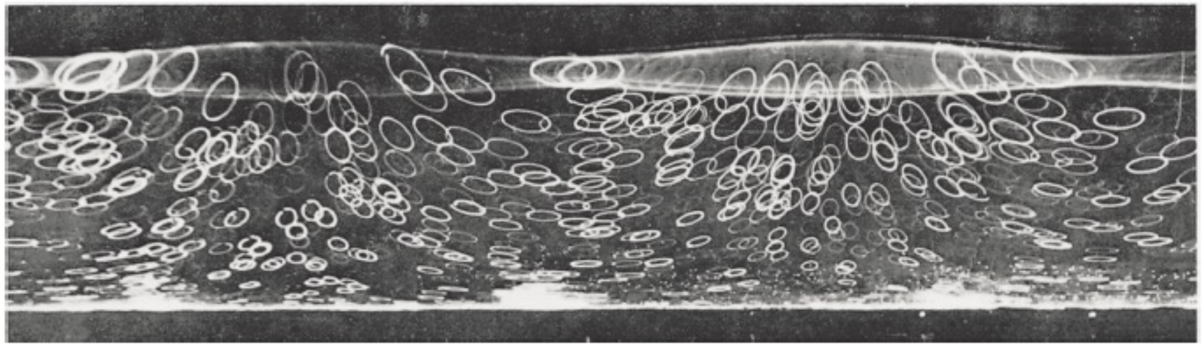
Description of Fluid Motion

Pathlines, Streaklines, and Streamlines

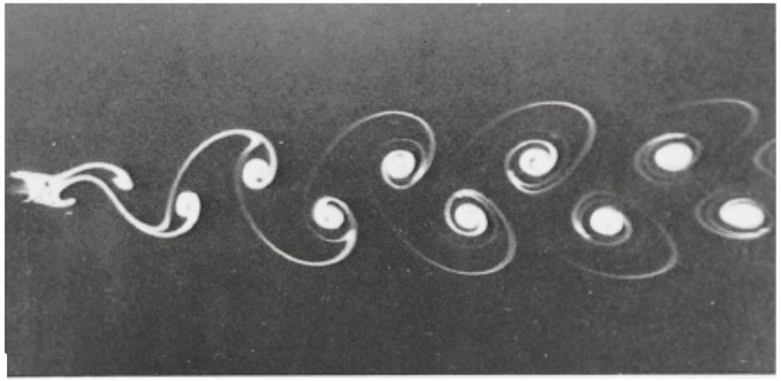
- **Pathline**: Locus of points traversed by a given particle as it travels in a field of flow.
- **Streakline**: An instantaneous line whose points are occupied by all particles from a specified point.
- **Streamline**: Line in the flow where the velocity vector of each particle on the streamline is tangent to the streamline.

Description of Fluid Motion

Pathlines, Streaklines, and Streamlines



Pathlines underneath a wave in a tank of water. (Photograph by A. Wallet and F. Ruellan. Courtesy of M. C. Vasseur.)



Streaklines in the unsteady flow around a cylinder. (Photograph by Sadatoshi Taneda. From Album of Fluid Motion, 1982, The Parabolic Press, Stanford, California.)

Description of Fluid Motion

Pathlines, Streaklines, and Streamlines

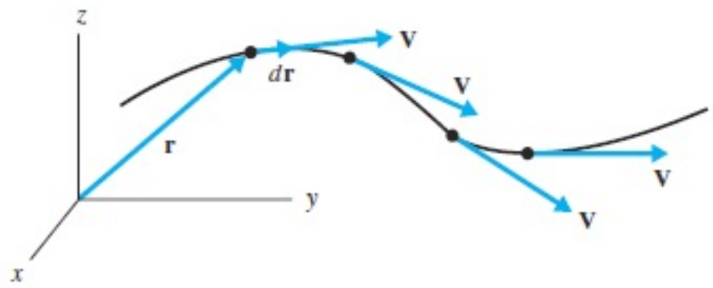


Figure 3.3 Streamline in a flow field.

- A streamline can be expressed as:

$$\mathbf{V} \times d\mathbf{r} = 0$$

- As *V* and *dr* are in the same direction, the cross-product of these vectors is zero.

Description of Fluid Motion

Pathlines, Streaklines, and Streamlines

- **Streamtube**: A tube whose walls are streamlines.
  - As velocity is tangent to a streamline, no fluid crosses the walls of a streamtube.
  - E.g., Pipes or open channels.

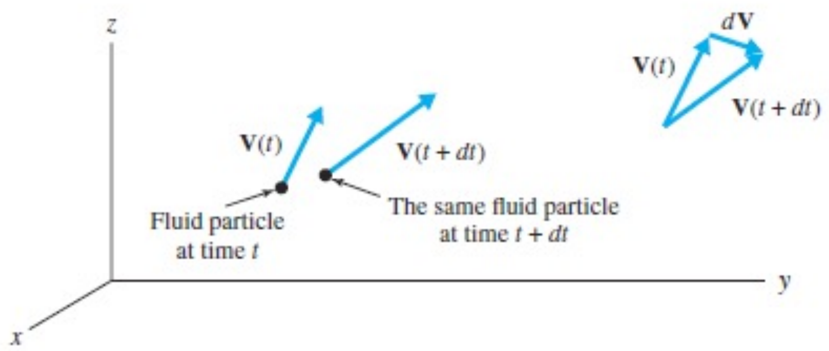
- **In a steady flow, pathlines, streaklines, and streamlines are all coincident.**

Description of Fluid Motion

Acceleration

- Acceleration is the derivative of velocity (with respect to time).

$$\mathbf{V} = u\hat{\mathbf{i}} + v\hat{\mathbf{j}} + w\hat{\mathbf{k}}$$



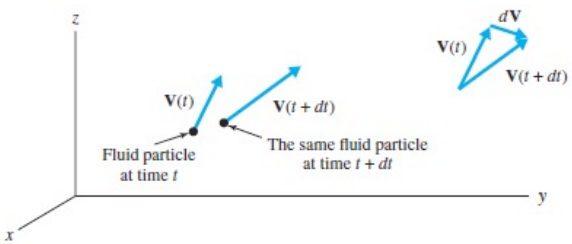
Velocity of a fluid particle.

$$\mathbf{a} = \frac{\partial \mathbf{V}}{\partial x}\frac{dx}{dt} + \frac{\partial \mathbf{V}}{\partial y}\frac{dy}{dt} + \frac{\partial \mathbf{V}}{\partial z}\frac{dz}{dt} + \frac{\partial \mathbf{V}}{\partial t}$$

$$\frac{dx}{dt} = u \qquad \frac{dy}{dt} = v \qquad \frac{dz}{dt} = w$$

Description of Fluid Motion

Acceleration



Velocity of a fluid particle.

- The acceleration is:

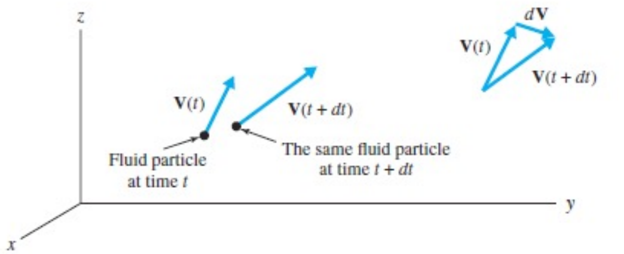
$$\mathbf{a} = u\frac{\partial \mathbf{V}}{\partial x} + v\frac{\partial \mathbf{V}}{\partial y} + w\frac{\partial \mathbf{V}}{\partial z} + \frac{\partial \mathbf{V}}{\partial t}$$

- The scalar components of the above equation in rectangular coordinates are:

$$a_x = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$
$$a_y = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}$$
$$a_z = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}$$

Description of Fluid Motion

Acceleration



Velocity of a fluid particle.

- The acceleration equation can be simplified as follows:

$$\mathbf{a} = \frac{D\mathbf{V}}{Dt}$$

$$\frac{D}{Dt} = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} + \frac{\partial}{\partial t}$$

- The derivative is the “substantial derivative” or “material derivative.”

Description of Fluid Motion

Acceleration

- If the observer's reference frame is accelerating:
  - Acceleration of a particle relative to a fixed reference frame is needed.

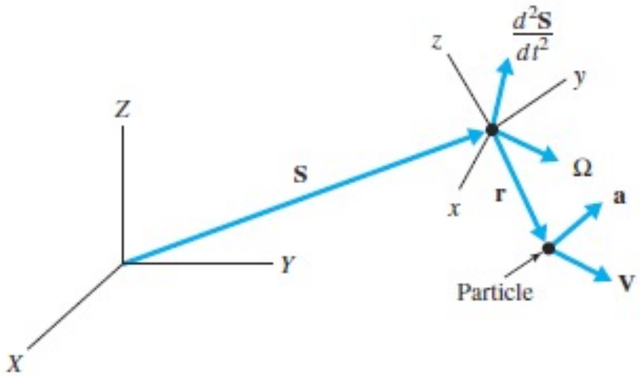
$$\mathbf{A} = \mathbf{a} + \underbrace{\frac{d^2\mathbf{S}}{dt^2}}_{\text{acceleration of reference frame}} + \underbrace{2\boldsymbol{\Omega} \times \mathbf{V}}_{\text{Coriolis acceleration}} + \underbrace{\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r})}_{\text{normal acceleration}} + \underbrace{\frac{d\boldsymbol{\Omega}}{dt} \times \mathbf{r}}_{\text{angular acceleration}}$$

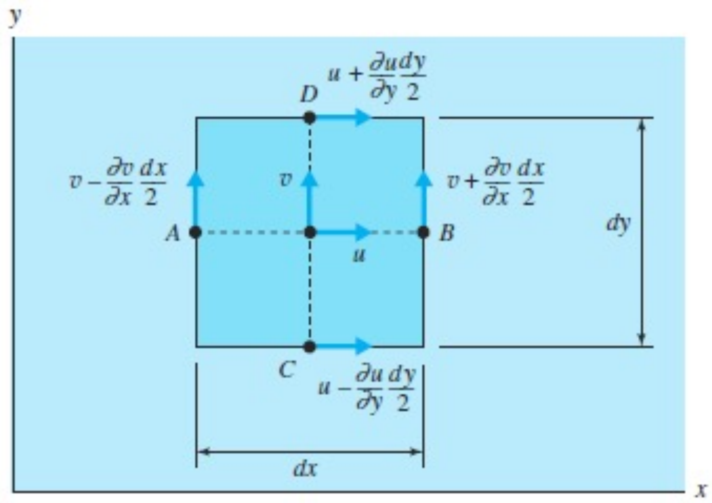
a: Acceleration given by the equation in a previous slide
V: Velocity vector of the particle
r: Position vector of the particle
Ω: Angular velocity of the observer's reference frame

- If A = a, the reference frame is **inertial**: a reference frame that moves with constant velocity without rotating.
- If A ≠ a, the reference frame is **noninertial**.

Description of Fluid Motion

Angular Velocity and Vorticity





- As a fluid particle moves it may rotate or deform.
  - In certain flows or regions, fluid particles do not rotate.
    - These are called **irrotational flows**.
    - E.g., Flow outside a thin boundary layer on airfoils, flow around submerged objects

Description of Fluid Motion

Angular Velocity and Vorticity

- **Angular Velocity (Ω)**: The average velocity of two perpendicular line segments of a fluid particle.

- **Vorticity (ω)**: Twice the angular velocity.
    - An irrotational flow has no vorticity.

Description of Fluid Motion

Angular Velocity and Vorticity

Substantial Derivative

Rectangular

$$\frac{D}{Dt} = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} + \frac{\partial}{\partial t}$$

Cylindrical

$$\frac{D}{Dt} = v_r\frac{\partial}{\partial r} + \frac{v_\theta}{r}\frac{\partial}{\partial \theta} + v_z\frac{\partial}{\partial z} + \frac{\partial}{\partial t}$$

Spherical

$$\frac{D}{Dt} = v_r\frac{\partial}{\partial r} + \frac{v_\theta}{r}\frac{\partial}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial}{\partial \phi} + \frac{\partial}{\partial t}$$

Vorticity

Rectangular

$$\omega_x = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \qquad \omega_y = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \qquad \omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

Cylindrical

$$\omega_r = \frac{1}{r}\left(\frac{\partial v_z}{\partial \theta}\right) - \frac{\partial v_\theta}{\partial z} \qquad \omega_\theta = \frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r} \qquad \omega_z = \frac{1}{r}\left(\frac{\partial (r v_\theta)}{\partial r} - \frac{\partial v_r}{\partial \theta}\right)$$

Spherical

$$\omega_r = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(v_\phi \sin\theta) - \frac{\partial v_\theta}{\partial \phi}\right] \qquad \omega_\phi = \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial \theta}\right]$$

$$\omega_\theta = \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r}(r v_\phi)\right]$$

Acceleration

Rectangular

$$a_x = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$

$$a_y = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}$$

$$a_z = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}$$

Cylindrical

$$a_r = \frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + v_z\frac{\partial v_r}{\partial z} - \frac{v_\theta^2}{r}$$

$$a_\theta = \frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + v_z\frac{\partial v_\theta}{\partial z} + \frac{v_r v_\theta}{r}$$

$$a_z = \frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z}$$

Spherical

$$a_r = \frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{v_\phi^2 + v_\theta^2}{r}$$

$$a_\theta = \frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_\phi}{r\sin\phi}\frac{\partial v_\theta}{\partial \phi} + \frac{v_r v_\theta - v_\phi^2\cot\theta}{r}$$

$$a_\phi = \frac{\partial v_\phi}{\partial t} + v_r\frac{\partial v_\phi}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\phi}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi} + \frac{v_r v_\phi + v_\theta v_\phi\cot\theta}{r}$$

Substantial Derivative, Acceleration, and Vorticity in Rectangular, Cylindrical, and Spherical Coordinates

Description of Fluid Motion

Angular Velocity and Vorticity

$$\Omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)$$

Angular Velocity about the x-axis

$$\Omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)$$

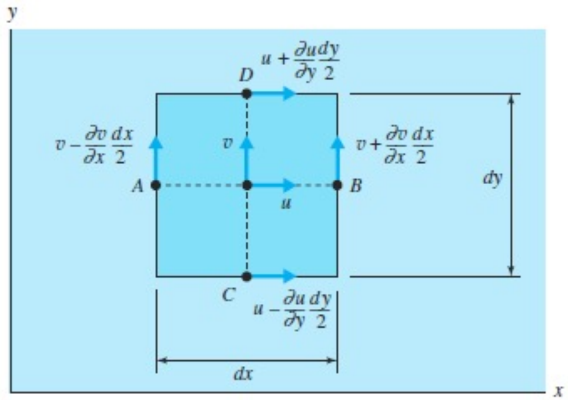
Angular Velocity about the y-axis

$$\Omega_z = \frac{1}{2}(\Omega_{AB} + \Omega_{CD})$$
$$= \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

Angular Velocity about the z-axis

Description of Fluid Motion

Angular Velocity and Vorticity



- **Rate-of-strain tensor (ε)**: The rate at which deformation occurs.
    - If AB rotates with an angular velocity different from that of CD, the particle deforms.

$$\epsilon_{xy} = \frac{1}{2}(\Omega_{AB} - \Omega_{CD}) = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

$$\epsilon_{xz} = \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)$$

$$\epsilon_{yz} = \frac{1}{2}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)$$

$$\epsilon_{xx} = \frac{u_B - u_A}{dx} = \left[u + \frac{\partial u}{\partial x}\frac{dx}{2} - \left(u - \frac{\partial u}{\partial x}\frac{dx}{2}\right)\right]\Big/ dx = \frac{\partial u}{\partial x}$$

$$\epsilon_{yy} = \frac{\partial v}{\partial y}$$

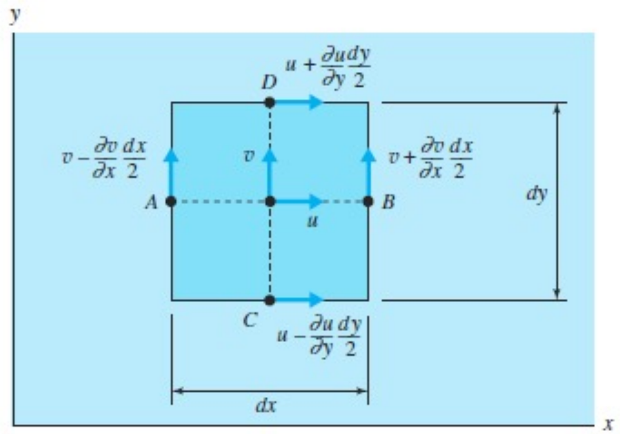
$$\epsilon_{zz} = \frac{\partial w}{\partial z}$$

- The complete tensor is shown below:

$$\epsilon_{ij} = \begin{pmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{xy} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{xz} & \epsilon_{yz} & \epsilon_{zz} \end{pmatrix}$$

Description of Fluid Motion

Angular Velocity and Vorticity



- The complete tensor is shown below:

$$\epsilon_{ij} = \begin{pmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{xy} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{xz} & \epsilon_{yz} & \epsilon_{zz} \end{pmatrix}$$

- This tensor is symmetric, i.e.:
  - $\varepsilon_{xy} = \varepsilon_{yx}$
  - $\varepsilon_{xz} = \varepsilon_{zx}$
  - $\varepsilon_{yz} = \varepsilon_{zy}$

Description of Fluid Motion

The velocity field is given by $\mathbf{V} = 2x\hat{\mathbf{i}} - yt\hat{\mathbf{j}}$ m/s, where x and y are in meters and t is in seconds. Find the equation of the streamline passing through (2, −1) and a unit vector normal to the streamline at (2, −1) at $t = 4$ s.

Solution

The velocity vector is tangent to a streamline so that $\mathbf{V} \times d\mathbf{r} = 0$ (the cross product of two parallel vectors is zero). For the given velocity vector we have, at $t = 4$ s,

$$\mathbf{V} \times d\mathbf{r} = (2x\hat{\mathbf{i}} - 4y\hat{\mathbf{j}}) \times (dx\hat{\mathbf{i}} + dy\hat{\mathbf{j}}) = (2x\,dy + 4y\,dx)\hat{\mathbf{k}} = 0$$

where we have used $\hat{\mathbf{i}} \times \hat{\mathbf{j}} = \mathbf{k}$, $\hat{\mathbf{j}} \times \hat{\mathbf{i}} = -\mathbf{k}$, and $\hat{\mathbf{i}} \times \hat{\mathbf{i}} = 0$. Consequently,

$$2x\,dy = -4y\,dx \qquad \text{or} \qquad \frac{dy}{y} = -2\frac{dx}{x}$$

Integrate both sides:

$$\ln y = -2\ln x + \ln C$$

where we used $\ln C$ for convenience. This is written as

$$\ln y = \ln x^{-2} + \ln C = \ln(Cx^{-2})$$

Hence

$$x^2 y = C$$

At (2, −1) we find $C = -4$, so that the streamline passing through (2, −1) has the equation

$$x^2 y = -4$$

A normal vector is perpendicular to the streamline, hence the velocity vector; using $\hat{\mathbf{n}} = n_x\hat{\mathbf{i}} + n_y\hat{\mathbf{j}}$ we have at (2, −1) and $t = 4$ s

$$\mathbf{V}\cdot\hat{\mathbf{n}} = (4\hat{\mathbf{i}} + 4\hat{\mathbf{j}})\cdot(n_x\hat{\mathbf{i}} + n_y\hat{\mathbf{j}}) = 0$$

Using $\hat{\mathbf{i}}\cdot\hat{\mathbf{i}} = 1$ and $\hat{\mathbf{i}}\cdot\hat{\mathbf{j}} = 0$, this becomes

$$4n_x + 4n_y = 0 \qquad \therefore n_x - n_y$$

Then, because $\hat{\mathbf{n}}$ is a unit vector, $n_x^2 + n_y^2 = 1$ and we find that

$$n_x^2 = 1 - n_x^2 \qquad \therefore n_x = \frac{\sqrt{2}}{2} \qquad \text{and} \qquad n_y = -\frac{\sqrt{2}}{2}$$

The unit vector normal to the streamline is written as

$$\hat{\mathbf{n}} = \frac{\sqrt{2}}{2}(\hat{\mathbf{i}} - \hat{\mathbf{j}})$$

Description of Fluid Motion

A velocity field in a particular flow is given by $\mathbf{V} = 20y^2\hat{\mathbf{i}} - 20xy\hat{\mathbf{j}}$ m/s. Calculate the acceleration, the angular velocity, the vorticity vector, and any nonzero rate-of-strain components at the point (1, −1, 2).

Solution

We could use Eq. 3.2.9 and find each component of the acceleration, or we could use Eq. 3.2.8 and find a vector expression. Using Eq. 3.2.8, we have

$$\begin{aligned} \mathbf{a} &= u\frac{\partial \mathbf{V}}{\partial x} + v\frac{\partial \mathbf{V}}{\partial y} + w\cancelto{0}{\frac{\partial \mathbf{V}}{\partial z}} + \cancelto{0}{\frac{\partial \mathbf{V}}{\partial t}} \\ &= 20y^2(-20y\hat{\mathbf{j}}) - 20xy(40y\hat{\mathbf{i}} - 20x\hat{\mathbf{j}}) \\ &= -800xy^2\hat{\mathbf{i}} - 400(y^3 - x^2y)\hat{\mathbf{j}} \end{aligned}$$

where we have used $u = 20y^2$ and $v = -20xy$, as given by the velocity vector. All particles passing through the point (1, −1, 2) have the acceleration

$$\mathbf{a} = \underline{-800\hat{\mathbf{i}}\ \text{m/s}^2}$$

The angular velocity has two zero components:

$$\Omega_x = \frac{1}{2}\left(\cancelto{0}{\frac{\partial w}{\partial y}} - \cancelto{0}{\frac{\partial v}{\partial z}}\right) = 0, \quad \Omega_y = \frac{1}{2}\left(\cancelto{0}{\frac{\partial u}{\partial z}} - \cancelto{0}{\frac{\partial w}{\partial x}}\right) = 0$$

The non-zero z-component, at the point (1, −1, 2) is

$$\begin{aligned} \Omega_z &= \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \\ &= \frac{1}{2}(-20y - 40y) = \underline{30\ \text{rad/s}} \end{aligned}$$

The vorticity vector is twice the angular velocity vector:

$$\boldsymbol{\omega} = 2\Omega_z\hat{\mathbf{k}} = \underline{60\hat{\mathbf{k}}\ \text{rad/s}}$$

The nonzero rate-of-strain components are

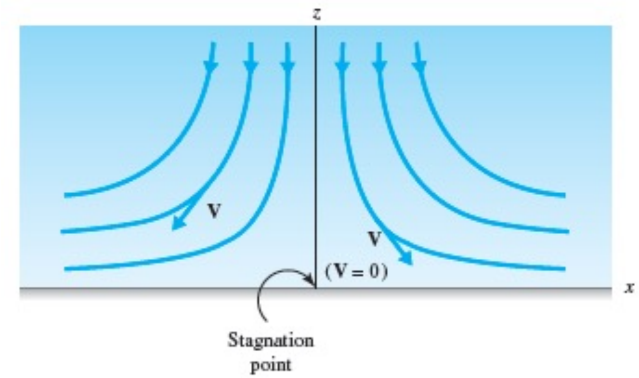
$$\begin{aligned} \epsilon_{xy} &= \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) \\ &= \frac{1}{2}(-20y + 40y) = \underline{-10\ \text{rad/s}} \\ \epsilon_{yy} &= \frac{\partial v}{\partial y} \\ &= -20x = \underline{-20\ \text{rad/s}} \end{aligned}$$

The rate-of-strain tensor would be written as

$$\epsilon_{ij} = \begin{bmatrix} 0 & -10 & 0 \\ -10 & -20 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{rad/s}$$

Classification of Fluid Flows

One-, Two-, and Three-Dimensional Flows



The figure above shows a three-dimensional flow with a stagnation point where the flow is normal to the plane surface

- **Three-dimensional flow**: The velocity vector depends on three spatial coordinates:

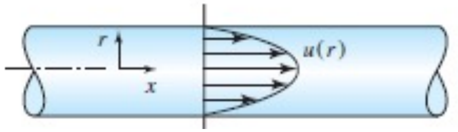
  $$V = V(x, y, z, t)$$

- **Stagnation point**: A point at which the fluid decelerates and comes to rest.

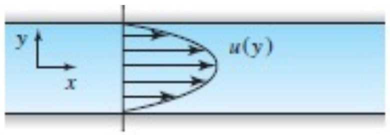
- A three-dimensional flow can often be approximated as a two-dimensional flow (a plane flow).

Classification of Fluid Flows

One-, Two-, and Three-Dimensional Flows



One-dimensional flow in a pipe u=u(r)



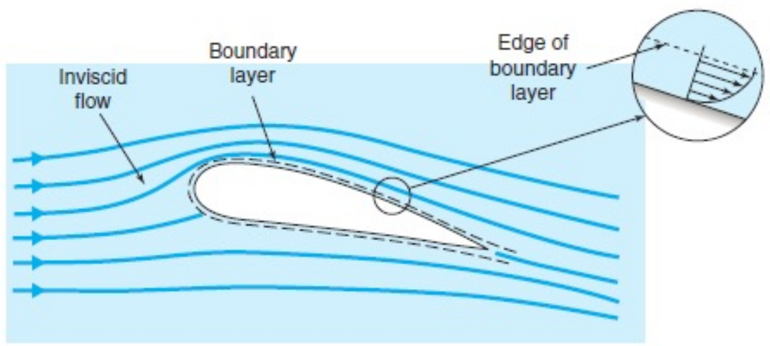
One-dimensional flow between parallel plates u=u(y)

- **One-dimensional flow**: The velocity vector depends on one variable.
- A **developed flow** is one in which the velocity profile doesn't vary with respect to the space coordinate in the direction of the flow.
- A **uniform flow** is one in which the fluid properties are constant over the area.
  - E.g., Relatively high speed flows in pipes, and flow in a stream.

Classification of Fluid Flows

Viscous and Inviscid Flows

- A fluid flow can either be a viscous flow or an inviscid flow.
  - **Inviscid flow**: Viscous effects do not significantly influence the flow.
  - **Viscous flow**: Effects of viscosity are important.

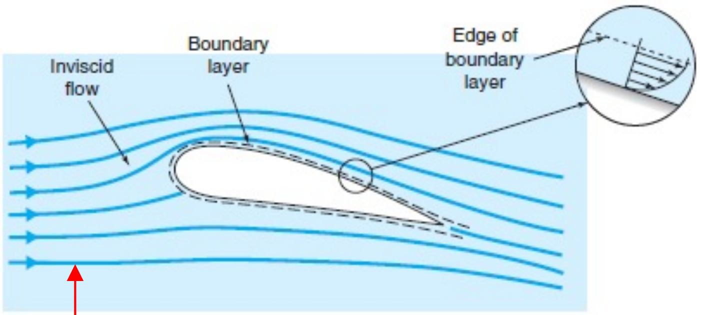


Flow around an airfoil.

- The main class of flows that can be modeled as inviscid flows are **external flows**.
  - Flows that exist exterior to bodies.
- Any viscous effects that (may) exist are confined to a thin **boundary layer**.
  - The velocity in this layer is always zero at a fixed wall (due to viscosity).

Classification of Fluid Flows

Viscous and Inviscid Flows



Flow around an airfoil.

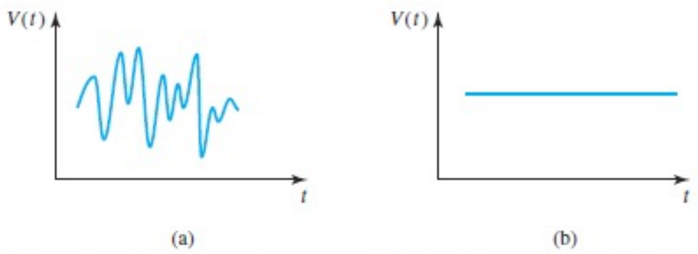
The inviscid flow outside the boundary layer in an external flow is called the **free stream**.

- Most of the time, boundary layers are very thin.
  - Can be ignored when studying flow around a streamlined body.
- Viscous flows include internal flows (flows in pipes and conduits and in open channels).
  - Create losses and accounts for huge amounts of energy for oil/gas transportation in pipelines.
  - No-slip condition results in zero velocity at the wall, and the resulting stresses lead to these losses.

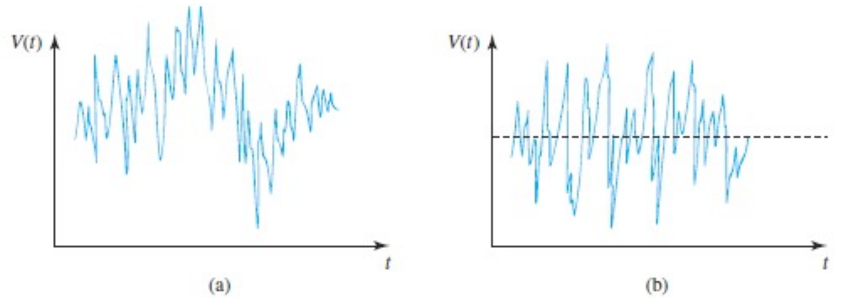
Classification of Fluid Flows

Laminar and Turbulent Flows

- Viscous flow is either laminar or turbulent.
  - **Laminar flow**: Flow with no significant mixing of particles but with significant viscous shear stresses.
  - **Turbulent flow**: Flow varies irregularly so that flow quantities (velocity/pressure) show random variation.
    - A "steady" turbulent flow is one in which the time-average physical quantities do not change in time.



Velocity as a function of time in a laminar flow: (a) unsteady flow; (b) steady flow.



Velocity as a function of time in a turbulent flow: (a) unsteady flow; (b) "steady" flow.

Classification of Fluid Flows

Laminar and Turbulent Flows

- Whether a flow is laminar or turbulent depends on three parameters:
  - Length scale of flow field
  - Velocity scale of the flow
  - Kinematic viscosity
- The **Reynolds Number** predicts the flow regime.

$$\mathrm{Re} = \frac{VL}{\upsilon}$$

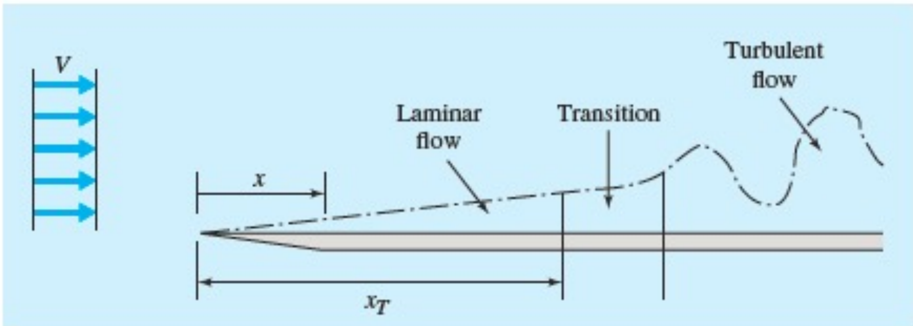
L: Characteristic Length
V: Characteristic Velocity
υ: Kinematic Viscosity

- If the Reynolds number is greater than the critical Reynolds number ($Re > Re_{crit}$) then the flow is turbulent {Laminar < Re_{crit} < Turbulent}

Classification of Fluid Flows

Laminar and Turbulent Flows

- If the Reynolds number is greater than the critical Reynolds number ($Re > Re_{crit}$) then the flow is turbulent:
  - Rough-walled pipe: $Re_{crit} \approx 2000$
  - Parallel plates: $Re_{crit} \approx 1500$
  - Flow on a flat plate: $Re_{crit} \approx 3 \times 10^5$



Boundary layer flow on a flat plate.

Classification of Fluid Flows

The 2-cm-diameter pipe of Figure E3.3 is used to transport water at 20°C. What is the maximum average velocity that may exist in the pipe for which laminar flow is guaranteed?

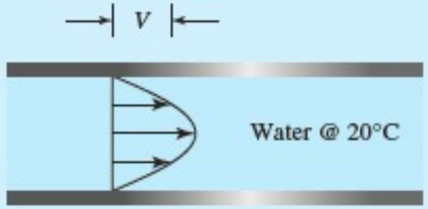


Figure E3.3

Solution

The kinematic viscosity is found in Appendix B to be $\nu = 10^{-6}$ m²/s, the value used if the temperature is not given. Using a Reynolds number of 2000 so that a laminar flow is guaranteed, we find that

$$V = \frac{2000\nu}{D}$$

$$= \frac{2000 \times 10^{-6}\ \text{m}^2/\text{s}}{0.02\ \text{m}} = 0.1\ \text{m/s}$$

This average velocity is quite small. Velocities this small are not usually encountered in actual situations; hence laminar flow is seldom of engineering interest except for specialized topics such as lubrication. Most internal flows are turbulent flows, and thus the study of turbulence gains much attention.

Classification of Fluid Flows

Incompressible and Compressible Flows

- Flows can be classified as either compressible or incompressible:
  - Incompressible flows are those in which the density of each fluid particle is constant. $\frac{D\rho}{Dt} = 0$

Classification of Fluid Flows

Incompressible and Compressible Flows

- **Mach Number**: A gas flow parameter that is the ratio of flow velocity to that of the speed of sound.

$$M = \frac{V}{c}$$

V: Gas Speed
c: Wave speed $c = \sqrt{kRT}$

- If M < 0.3, flow is assumed to be incompressible.

The Bernoulli Equation

- The Bernoulli equation states that for an inviscid fluid flow, an increase in fluid velocity causes a decrease in pressure or decrease in the potential energy of the fluid.

$$\frac{V^2}{2} + \frac{p}{\rho} + gh = \text{const}$$

Between two points on the same streamline:

$$\frac{V_1^2}{2} + \frac{p_1}{\rho} + gh_1 = \frac{V_2^2}{2} + \frac{p_2}{\rho} + gh_2$$

Assumptions

- Inviscid flow (no shear stress)
- Steady flow $\frac{\partial V}{\partial t} = 0$
- Along a streamline
- Constant density
- Inertial reference frame

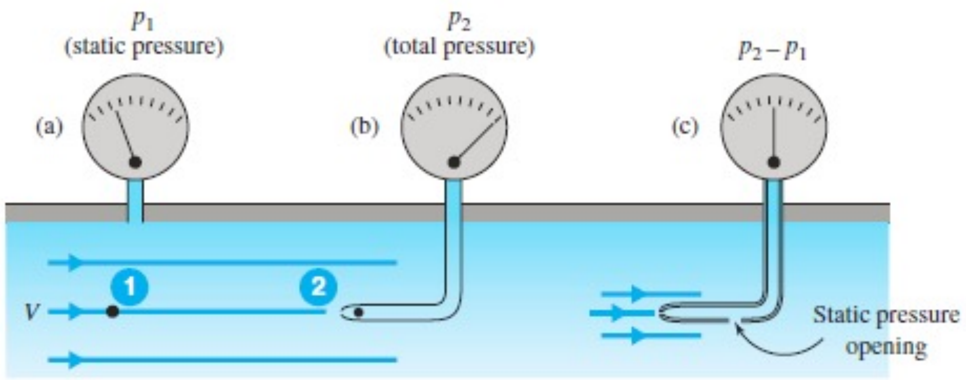
The Bernoulli Equation

- Another form of the equation (by dividing by g) is:

$$\frac{V_1^2}{2g} + \frac{p_1}{\gamma} + h_1 = \frac{V_2^2}{2g} + \frac{p_2}{\gamma} + h_2$$

1. Pressure p, is called the **static pressure (gage pressure)**.
2. Piezometric head is $\frac{p}{\gamma} + h$ and the total head is $\frac{p}{\gamma} + h + \frac{V^2}{2g}$
3. The total pressure at a stagnation point (local fluid velocity is zero) is the **Stagnation pressure**. $p + \rho\frac{V^2}{2} = p_T$

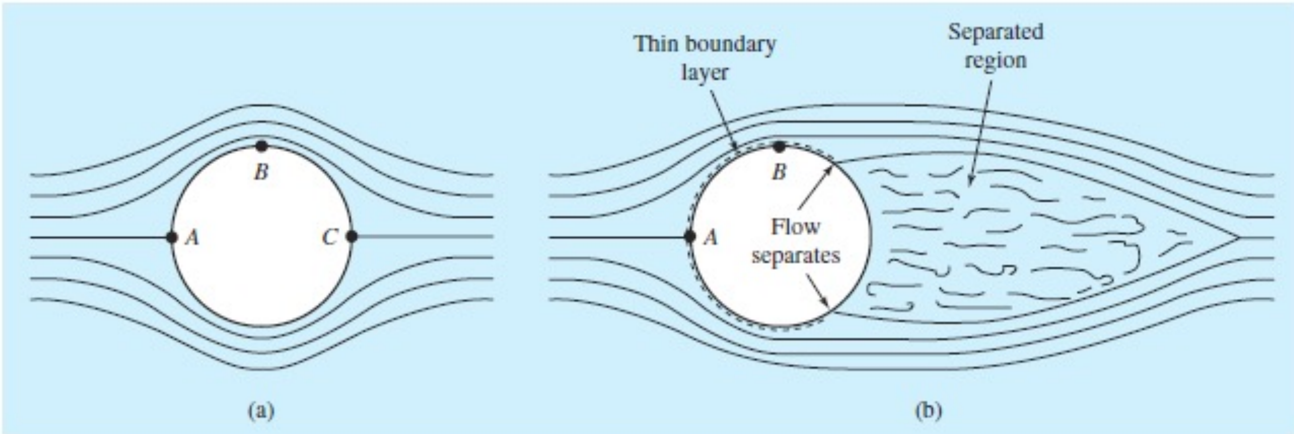
The Bernoulli Equation



Pressure probes: (a) piezometer; (b) pitot probe; (c) pitot-static probe.

1. A piezometer (left) is used to measure static pressure.
2. A pitot probe (center) is used to measure total pressure.
   a) Point 2 is a stagnation point.
3. A pitot-static probe (right) is used to measure the difference between total and static pressure.

The Bernoulli Equation



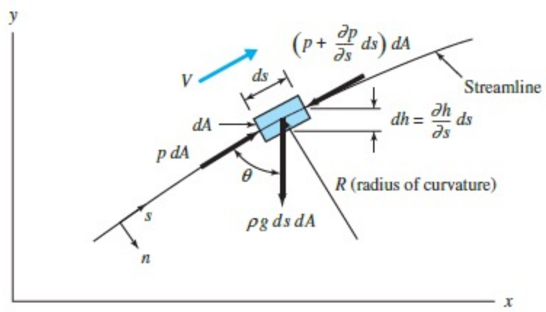
Flow around a sphere at relatively large Reynolds numbers: (a) inviscid flow; (b) actual flow.

- Maximum pressure at stagnation points A and C (velocity is zero).
- Maximum velocity (minimum pressure) at point B.

- In an actual flow, near C, the boundary streamline leaves the boundary (separated region).
  - Pressure remains relatively low at the rear.
  - Leads to a relatively large drag force in the direction of the flow.

The Bernoulli Equation

$$\Delta p = -\rho \frac{V^2}{R} \Delta n$$



- The equation above shows how the pressure changes normal to the streamline.
  - Δp: Incremental pressure change
  - Δn: Short distance
  - R: Radius of curvature

- Pressure decreases in the n-direction.
- Decrease is directly proportional to ρ and V^2
- Decrease is inversely proportional to R

The Bernoulli Equation

The wind reaches a speed of 144 km/h in a storm. Calculate the force acting on the 0.9 m × 1.8 m window of Figure E3.4 facing the storm. The window is in a high-rise building, so the wind speed is not reduced due to ground effects. Use $\rho = 1.27\ \text{kg/m}^3$.

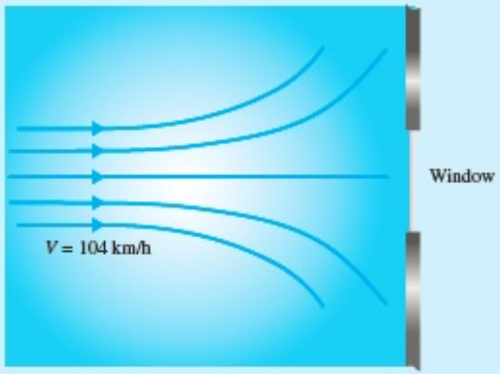


Figure E3.4

Solution

The window facing the storm will be in a stagnation region where the wind speed is brought to zero. Working with gage pressures, the pressure p upstream in the wind is zero. The velocity V must have units of m/s. It is

$$V = \frac{144 \times 10^3}{3600}\frac{\text{m}}{\text{s}} = 40\ \text{m/s}$$

Bernoulli's equation can be used in this situation since we can neglect viscous effects, and steady flow occurs along a streamline at constant density (air is incompressible at speeds below about 350 km/h). We calculate the pressure on the window selecting state 1 in the free stream and state 2 on the window, as follows:

$$\frac{V_1^2}{2g} + \cancel{\frac{p_1}{\gamma}} + \cancel{h_1} = \cancel{\frac{V_2^2}{2g}} + \frac{p_2}{\gamma} + \cancel{h_2}$$

$$\therefore\ p_2 = \frac{\rho V_1^2}{2}$$

$$= \frac{1.27\ \text{kg/m}^3 \times 40^2\ \text{m}^2/\text{s}^2}{2000} = 1.016\ \text{kPa}$$

where we have used $\gamma = \rho g$, $h_2 = h_1$, $p_1 = 0$, and $V_2 = 0$. Multiply by the area and find the force to be

$$F = pA$$
$$= 1.016 \times 0.9 \times 1.8 = \underline{1.646\ \text{kN}}$$

We recommend that you verify the units of N/m² on the pressure calculation above.

The Bernoulli Equation

The static pressure head in an air pipe (Figure E3.5) is measured with a piezometer as 16 mm of water. A pitot probe indicates 24 mm of water. Calculate the velocity of the 20°C air. Also, calculate the Mach number and comment as to the compressibility of the flow.

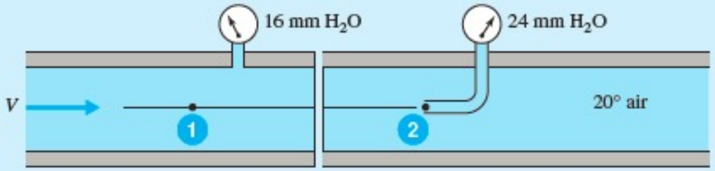


Figure E3.5

Solution

Bernoulli's equation is applied between two points on the streamline that terminates at the stagnation point of the pitot probe. Point 1 is upstream and p_2 is the total pressure at point 2; then, with no elevation change,

$$\frac{V_1^2}{2g} + \frac{p_1}{\gamma} = \frac{p_T}{\gamma}$$

The pressure measured with the piezometer is $p_1 = \gamma h = 9810 \times 0.016 = 157$ Pa. We use the ideal gas law to calculate the density:

$$\rho = \frac{p}{RT}$$

$$= \frac{(157 + 101\,000)\,\text{Pa}}{287\,\text{kJ/kg}\cdot\text{K} \times (273 + 20)\,\text{K}} = 1.203\,\text{kg/m}^3$$

where standard atmospheric pressure, which is 101 000 Pa (if no elevation is given, assume standard conditions), is added since absolute pressure is needed in the ideal gas law. The velocity is then

$$V_1 = \sqrt{\frac{2}{\rho}(p_T - p_1)}$$

$$= \sqrt{\frac{2(0.024 \times 9810 - 157)\,\text{Pa}}{1.203\,\text{kg/m}^3}} = \underline{11.42\text{ m/s}}$$

To find the Mach number, we must calculate the speed of sound. From Eq. 1.7.17 it is

$$c = \sqrt{kRT}$$

$$= \sqrt{1.4 \times 287\,\text{kJ/kg}\cdot\text{K} \times 293\,\text{K}} = 343\,\text{m/s}$$

The Mach number is then

$$\text{M} = \frac{V}{c} = \frac{11.44}{343} = \underline{0.0334}$$

Obviously, the flow can be assumed to be incompressible since $\text{M} < 0.3$. The velocity would have to be much higher before compressibility would be significant.

The Bernoulli Equation

Bernoulli's equation, in the form of Eq. 3.4.8, looks very much like the energy equation developed in thermodynamics for a control volume. Discuss the differences between the two equations.

Solution

From thermodynamics we recall that the steady-flow energy equation for a control volume with one inlet and one outlet takes the form

$$\dot{Q} - \dot{W}_s = \dot{m}\left(\frac{V_2^2}{2} + \frac{p_2}{\rho_2} + \tilde{u}_2 + gz_2\right) - \dot{m}\left(\frac{V_1^2}{2} + \frac{p_1}{\rho_1} + \tilde{u}_1 + gz_1\right)$$

This becomes, after dividing through by g,

$$\frac{V_2^2}{2g} + \frac{p_2}{\gamma} + z_2 = \frac{V_1^2}{2g} + \frac{p_1}{\gamma} + z_1$$

where we have made the following assumptions:

No heat transfer ($\dot{Q} = 0$)
No shaft work ($\dot{W}_s = 0$)
No temperature change ($\tilde{u}_2 = \tilde{u}_1$, i.e., no losses due to shear stresses)
Uniform velocity profiles at the two sections
Steady flow
Constant density ($\gamma_2 = \gamma_1$)

Even though several of these assumptions are the same as those made in the derivation of the Bernoulli equation (steady flow, constant density, and no shear stress), we must not confuse the two equations; the Bernoulli equation is derived from Newton's second law and is valid along a streamline, whereas the energy equation is derived from the first law of thermodynamics and is valid between two sections in a fluid flow. The energy equation can be used across a pump to determine the horsepower required to provide a particular pressure rise; the Bernoulli equation can be used along a stagnation streamline to determine the pressure at a stagnation point, a point where the velocity is zero. The equations are quite different, and just because the energy equation degenerates to the Bernoulli equation for particular situations, the two should not be used out of context.

The Bernoulli Equation

Explain why a burr on the upstream side of the piezometer opening of Figure 3.18a will result in a low reading of the pressure.

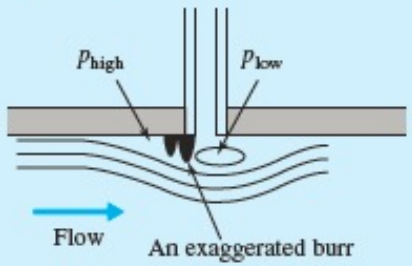


Figure E3.7

Solution

A burr on the upstream side of the piezometer opening would result in a flow in the vicinity of the burr somewhat like that shown in Figure E3.7. A streamline pattern would develop so that a relatively high pressure would occur on the upstream side of the burr and a relatively low pressure on the downstream side at the opening of the piezometer tube. Consequently, since the center of the curvature of the streamline is in the vicinity of the opening, a lower reading of the pressure would be recorded. If the burr were on the downstream side of the opening, a higher pressure reading would be recorded.

Summary

- The Eulerian description for acceleration is:

$$\mathbf{a} = u\frac{\partial \mathbf{V}}{\partial x} + v\frac{\partial \mathbf{V}}{\partial y} + w\frac{\partial \mathbf{V}}{\partial z} + \frac{\partial \mathbf{V}}{\partial t}$$

- For a flow in the xy-plane, a particle rotates with angular velocity:

$$\Omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

- This deforms with:

$$\epsilon_{xx} = \frac{\partial u}{\partial x}, \quad \epsilon_{yy} = \frac{\partial v}{\partial y}, \quad \epsilon_{xy} = \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$$

Summary

- Fluid flows can either be:
  - Steady or unsteady
  - Viscous or inviscid
  - Laminar, turbulent or free-stream
  - Incompressible or compressible
- The Bernoulli equation is :

$$\frac{V_1^2}{2g} + \frac{p_1}{\gamma} + h_1 = \frac{V_2^2}{2g} + \frac{p_2}{\gamma} + h_2$$

For a steady, inviscid, constant-density flow along a streamline in an inertial reference frame

- The pressure change normal to a streamline is:

$$\Delta p = -\rho \frac{V^2}{R} \Delta n$$