Flow in Pipe

Introduction

Study of viscosity on an incompressible, internal flow.

E.g., Flow in a circular pipe

$$\mathrm{Re} = \frac{V\rho l}{\mu}$$

- The Reynolds number is a ratio of inertial forces to viscous forces.
  - Important when dealing with viscous effects in a flow.

- When the ratio is large, inertial forces dominate viscous forces.
  - True when there are short, sudden geometric changes.
  - Viscous effects are important when surface areas are large.

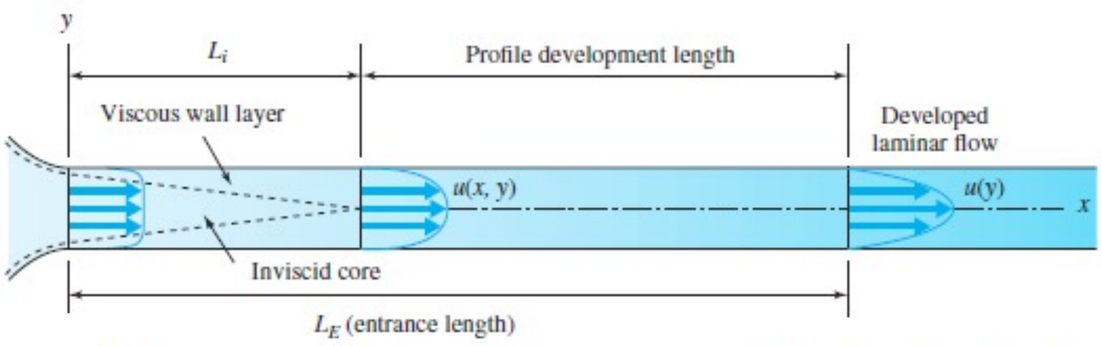
Introduction

$$\mathrm{Re} = \frac{V\rho l}{\mu}$$

- **Laminar Flow**:
  - Re < 2000 for pipes
  - Re < 1500 in a wide channel

- At a sufficiently high Reynolds number, a turbulent flow occurs.

Entrance Flow and Developed Flow

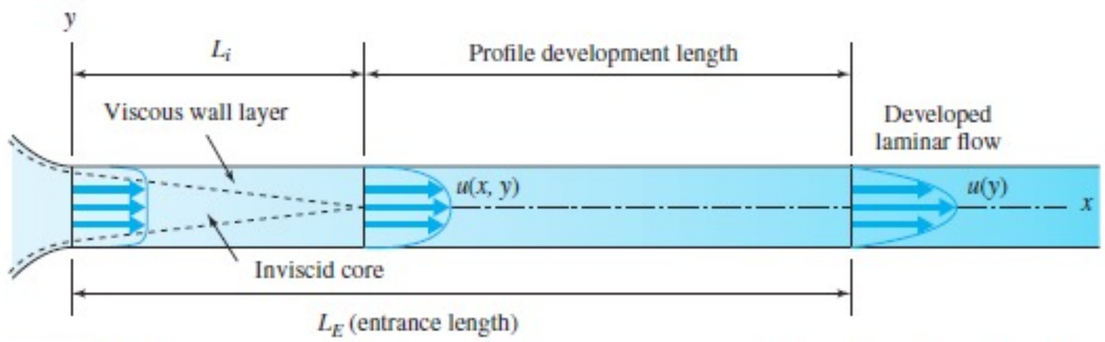


Entrance region of a laminar flow in a pipe or a wide rectangular channel.

- **Developed Laminar Flow** - Flow where the velocity profile ceases to change in the flow direction.

- In the entrance region of a laminar flow, the velocity profile changes in the flow direction.
- Idealized flow from a reservoir begins at the inlet as a uniform flow.
- Viscous wall layer grows over the inviscid core length, L_i until the viscous stresses dominate the entire cross section.

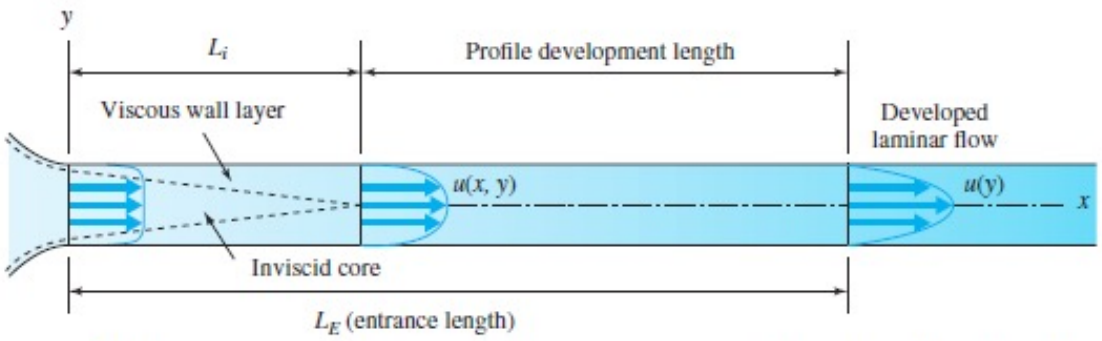
Entrance Flow and Developed Flow



Entrance region of a laminar flow in a pipe or a wide rectangular channel.

- The profile develops due to viscous effects until a developed flow is achieved.
  - **The inviscid core length is one-fourth to one-third of the entrance length L_E.**
  - This depends on the conduit geometry, shape of the inlet velocity profile, and the Reynolds number.

Entrance Flow and Developed Flow



Entrance region of a laminar flow in a pipe or a wide rectangular channel.

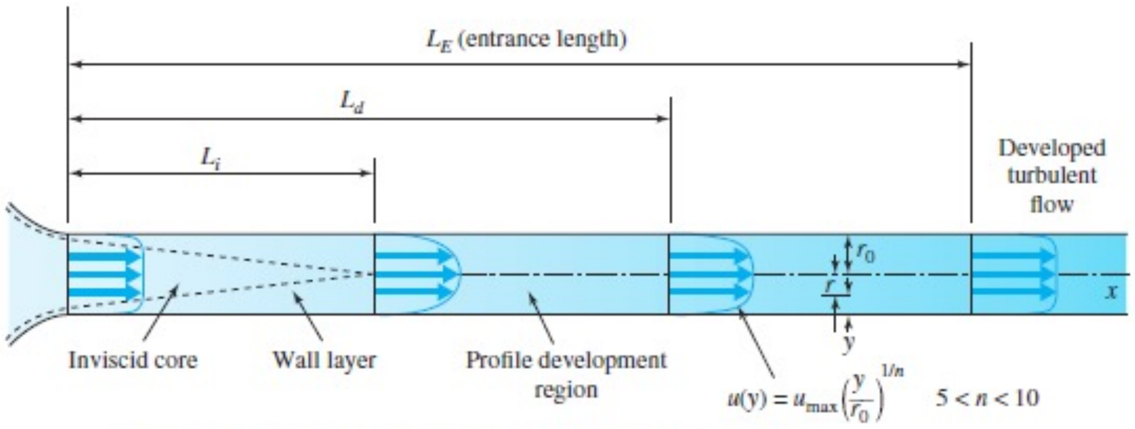
$$\frac{L_E}{D} = 0.065\,\mathrm{Re} \qquad \mathrm{Re} = \frac{VD}{\nu}$$

The entrance length equation for a **laminar flow in a circular pipe** with a uniform profile at the inlet. Re_{lam} = 2000

$$\frac{L_E}{h} = 0.04\,\mathrm{Re} \qquad \mathrm{Re} = \frac{Vh}{\nu}$$

For a **laminar flow in a high-aspect-ratio channel** with a uniform profile at the inlet. Re_{lam} = 1500

Entrance Flow and Developed Flow

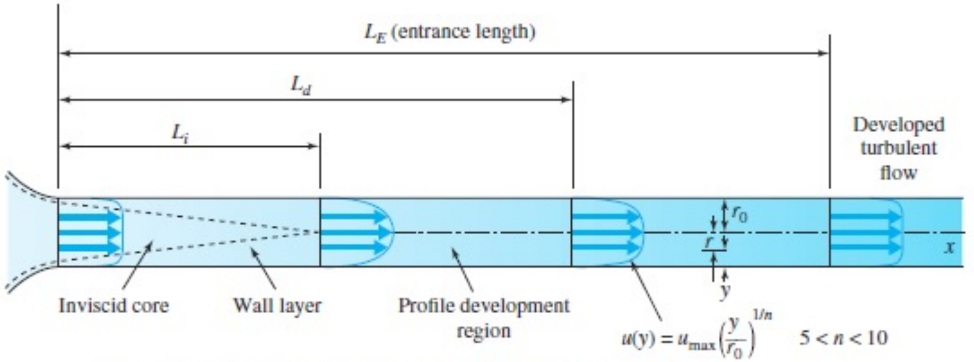


Velocity profile development in a turbulent pipe flow.

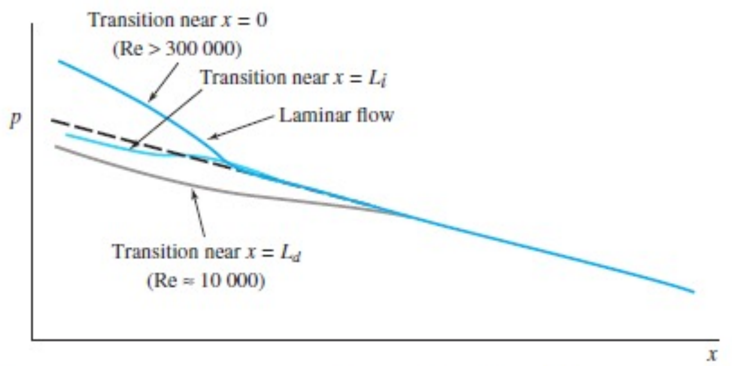
- **For a large Reynolds number (Re > 10^5)**

$$\frac{L_i}{D} \simeq 10 \qquad \frac{L_d}{D} \simeq 40 \qquad \frac{L_E}{D} \simeq 120$$

Entrance Flow and Developed Flow



Velocity profile development in a turbulent pipe flow.

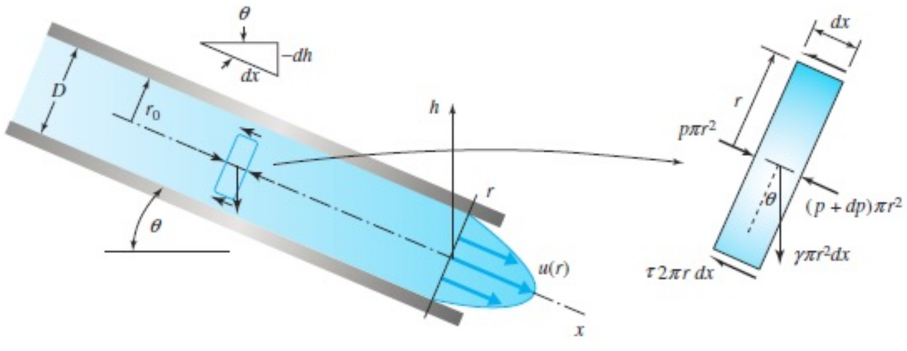


Pressure variation in a horizontal pipe flow for both laminar and turbulent flows. (From PhD Thesis of Dr. Jack Backus, Michigan State University)

- For a flow beyond a large x, the pressure variation decreases linearly with x.
- Transition near origin→ Linear pressure variation begins near L_i → Pressure gradient in the inlet is higher than in the developed flow region.
- Transition near L_d→ (Low Re)→ Linear variation begins at the end of the transition→ Pressure gradient in the inlet is less than that of developed flow.
- Laminar flow→ Pressure variation is the same as that from a large Reynolds number→ Pressure gradient is higher than in the developed flow region.

Laminar Flow in a Pipe

Elemental Approach



Developed flow in a circular pipe.

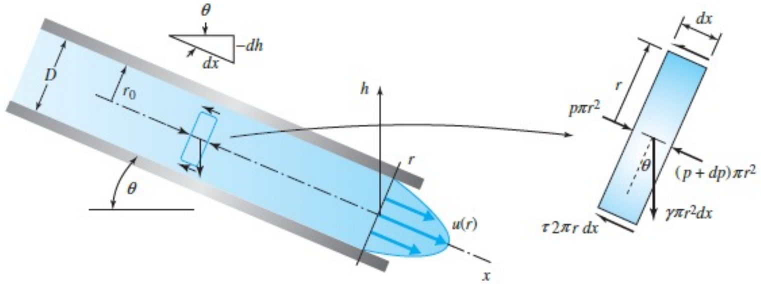
To investigate incompressible, steady, developed laminar flow in a pipe

- **Elemental approach:**
  - Infinitesimal control volume into which and from which fluid flows [Use momentum equation].
  - Infinitesimal fluid mass upon which forces act [Use Newton's second law].

- Since velocity profile doesn't change in the x-direction:
  - Momentum Flux in = Momentum Flux out and the resultant force is zero
  - No acceleration of the mass element; resultant force is zero.

Laminar Flow in a Pipe

Elemental Approach



$$u(r) = \frac{1}{4\mu}\frac{d(p + \gamma h)}{dx}\left(r^2 - r_0^2\right)$$

- The velocity distribution is **parabolic**.
- Called a **Poiseuille flow**.
  - This is a laminar flow with a parabolic profile in a pipe or between parallel plates.

Laminar Flow in a Pipe

Solving the Navier-Stokes Equations

$$\rho\left(\underbrace{\cancel{\frac{\partial u}{\partial t}}}_{\text{steady}} + \underbrace{\cancel{v_r}}_{\text{streamlines} \parallel \text{wall}}\frac{\partial u}{\partial r} + \underbrace{\cancel{\frac{v_\theta}{r}}}_{\text{no swirl}}\frac{\partial u}{\partial \theta} + u\underbrace{\cancel{\frac{\partial u}{\partial x}}}_{\text{developed flow}}\right)$$

$$= -\frac{\partial p}{\partial x} + \gamma\sin\theta + \mu\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \underbrace{\frac{1}{r^2}\cancel{\frac{\partial^2 u}{\partial \theta^2}}}_{\text{symmetric flow}} + \underbrace{\cancel{\frac{\partial^2 u}{\partial x^2}}}_{\text{developed flow}}\right)$$

Navier-Stokes equation for flow in a circular pipe:

- Developed flow
- Streamlines are parallel to the wall
- No swirl
- No acceleration of the fluid particles as they move in the pipe.

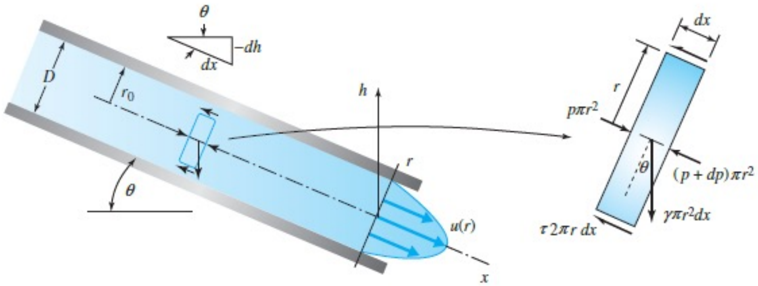
$$u(r) = \frac{\lambda}{4}(r^2 - r_0^2)$$

$$= \frac{1}{4\mu}\frac{d(p+\gamma h)}{dx}(r^2 - r_0^2)$$

This **parabolic** velocity distribution for flow in a pipe is called a **Poiseuille flow**.

Laminar Flow in a Pipe

Pipe Flow Quantities



- For steady, laminar, developed flow in a circular pipe, the velocity distribution is:

$$u(r) = \frac{1}{4\mu}\frac{d(p+\gamma h)}{dx}\left(r^2 - r_0^2\right)$$

- The average velocity, V, is:

$$V = \frac{Q}{A} = \frac{1}{\pi r_0^2}\int_0^{r_0} u(r)\,2\pi r\,dr$$

$$= \frac{2}{r_0^2}\int_0^{r_0}\frac{1}{4\mu}\frac{d(p+\gamma h)}{dx}\left(r^2 - r_0^2\right)r\,dr = -\frac{r_0^2}{8\mu}\frac{d(p+\gamma h)}{dx}$$

- For a horizontal pipe, the pressure drop is as follows:

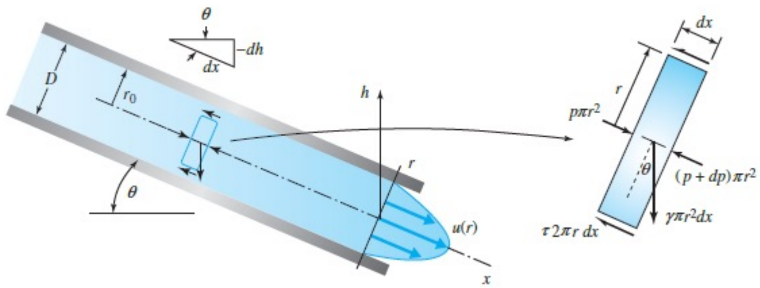
$$\Delta p = \frac{8\mu VL}{r_0^2}$$

For an inclined pipe, p is replaced with (p+γh)

$$\Delta p = \frac{2\tau_0 L}{r_0}$$

Laminar Flow in a Pipe

Pipe Flow Quantities



- The **friction factor, f** :
  - Dimensionless wall shear valid for both laminar and turbulent flow.

$$f = \frac{\tau_0}{\frac{1}{8}\rho V^2}$$

$$f = \frac{64}{\mathrm{Re}}$$

For laminar flow in a pipe

- Also, head loss (h_L) with a dimension of length:

$$\frac{\Delta p}{\gamma} = h_L = f\,\frac{L}{D}\frac{V^2}{2g}$$

Darcy-Weisbach equation (valid for both laminar and turbulent flows).

$$h_L = \frac{32\mu L V}{\gamma D^2}$$

Head-loss is directly proportional to the average velocity in a laminar flow.

Laminar Flow in a Pipe

A small-diameter horizontal tube is connected to a supply reservoir as shown in Figure E7.1. If 6600 mm^3 is captured at the outlet in 10 s, estimate the viscosity of the water.

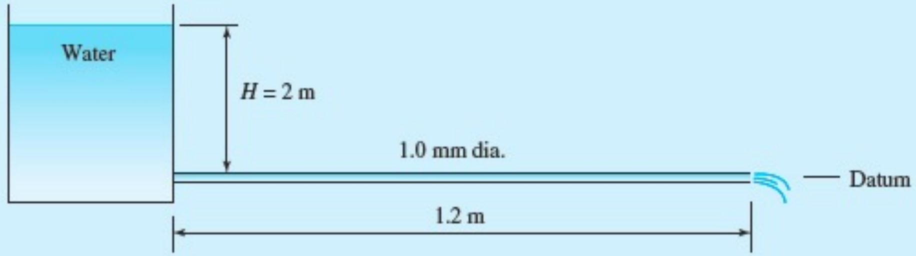


Figure E7.1

Solution

The tube is very small, so we expect viscous effects to limit the velocity to a small value. Using Bernoulli's equation from the surface to the entrance to the tube, and neglecting the velocity head, we have, letting 0 be a point on the reservoir surface,

$$\frac{\cancel{p_0}^{\,0}}{\cancel{\gamma}} + H = \frac{\cancel{V^2}^{\,0}}{\cancel{2g}} + \frac{p}{\gamma}$$

where we have used gage pressure with $p_0 = 0$. This becomes, assuming $V^2/2g \cong 0$ at the tube's entrance,

$$p = \gamma H = 9800 \text{ N/m}^2 \times 2 \text{ m} = 19\,600 \text{ Pa}$$

At the exit of the tube the pressure is zero; hence

$$\frac{\Delta p}{L} = \frac{19\,600}{1.2} = 16\,300 \text{ Pa/m (N/m}^3)$$

Laminar Flow in a Pipe

The average velocity is found to be

$$V = \frac{Q}{A} = \frac{6600 \times 10^{-9}/10}{\pi \times 0.001^2/4} = 0.840 \text{ m/s}$$

Check to make sure the velocity head is negligible: $V^2/2g = 0.036$ m compared with $p/\gamma = 2$ m, so the assumption of negligible velocity head is valid and our pressure calculation is acceptable. Using Eq. 7.3.14, we can find the viscosity of this assumed laminar flow to be

$$\mu = \frac{r_0^2}{8V}\frac{\Delta p}{L} = \frac{0.0005^2 \text{ m}^2}{8 \times 0.84 \text{ m/s}}\left(16\,300 \text{ N/m}^3\right) = \underline{6.06 \times 10^{-4} \text{ N} \cdot \text{s/m}^2}$$

We should check the Reynolds number to determine if our assumption of a laminar flow is acceptable. It is

$$\text{Re} = \frac{\rho V D}{\mu} = \frac{1000 \text{ kg/m}^3 \times 0.84 \text{ m/s} \times 0.001 \text{ m}}{6.06 \times 10^{-4} \text{ N} \cdot \text{s/m}^2} = 1390$$

where we use $\text{kg/m}^3 = \text{N} \cdot \text{s}^2/\text{m}^4$. This is obviously a laminar flow since $\text{Re} < 2000$, so the calculations are valid providing the entrance length is not too long. It is

$$L_E = 0.068 \text{Re} \times D = 0.065 \times 1390 \times 0.001 = 0.09 \text{ m}$$

This is approximately 8% of the total length, a sufficiently small quantity; hence the calculation for viscosity is assumed acceptable.

Laminar Flow in a Pipe

Derive an expression for the velocity distribution and the flow rate between horizontal, concentric pipes for a steady, incompressible developed flow (Figure E7.2).

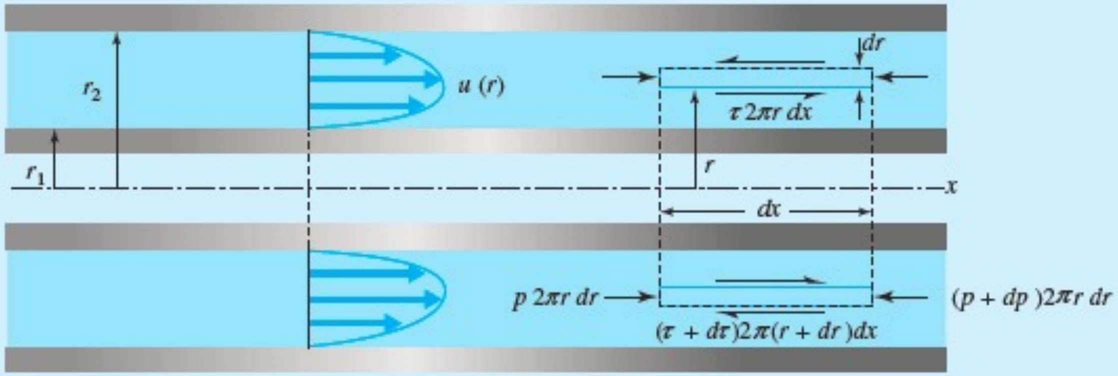


Figure E7.2

Solution

Let us use an elemental approach. The element is a hollow cylindrical shell as sketched in the figure. If we sum forces, we obtain

$$p2\pi r\, dr - (p + dp)2\pi r\, dr + \tau 2\pi r\, dx - (\tau + d\tau)2\pi(r + dr)dx = 0$$

Simplifying, there results, neglecting the term of differential magnitude,

$$\frac{dp}{dx} = -\frac{\tau}{r} - \frac{d\tau}{dr} - \underbrace{\cancel{\frac{d\tau}{r}}}_{\text{neglect}}$$

Laminar Flow in a Pipe

Substituting $\tau = -\mu\, du/dr$ (du/dr is negative near the outer wall where the element is sketched) we have

$$\frac{dp}{dx} = \mu\left(\frac{1}{r}\frac{du}{dr} + \frac{d^2u}{dr^2}\right)$$
$$= \frac{\mu}{r}\frac{d}{dr}\left(r\frac{du}{dr}\right)$$

Multiply both sides by rdr and divide by μ, then integrate:

$$r\frac{du}{dr} = \frac{1}{2\mu}\frac{dp}{dx}r^2 + A$$

Multiply both sides by dr/r and integrate again:

$$u(r) = \frac{1}{4\mu}\frac{dp}{dx}r^2 + A \ln r + B$$

where A and B are arbitrary constants. They are found by setting $u = 0$ at $r = r_1$ and at $r = r_2$; that is,

$$0 = \frac{1}{4\mu}\frac{dp}{dx}r_1^2 + A \ln r_1 + B$$
$$0 = \frac{1}{4\mu}\frac{dp}{dx}r_2^2 + A \ln r_2 + B$$

Laminar Flow in a Pipe

Solve for A and B:

$$A = \frac{1}{4\mu}\frac{dp}{dx}\frac{r_1^2 - r_2^2}{\ln(r_2/r_1)}$$

$$B = -A\ \ln\ r_2 - \frac{r_2^2}{4\mu}\frac{dp}{dx}$$

Thus

$$u(r) = \frac{1}{4\mu}\frac{dp}{dx}\left[r^2 - r_2^2 + \frac{r_2^2 - r_1^2}{\ln(r_1/r_2)}\ln(r/r_2)\right]$$

This is integrated to give the flow rate:

$$Q = \int_{r_1}^{r_2} u(r) 2\pi r\, dr$$

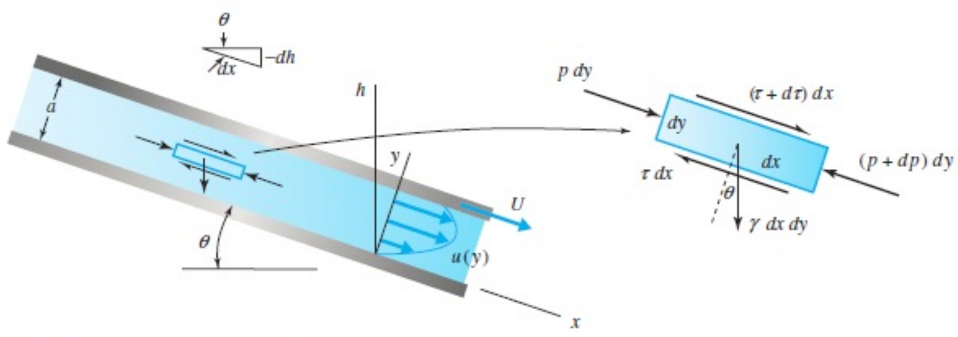
$$= -\frac{\pi}{8\mu}\frac{dp}{dx}\left[r_2^4 - r_1^4 - \frac{\left(r_2^2 - r_1^2\right)^2}{\ln(r_2/r_1)}\right]$$

As $r_1 \to 0$ the velocity distribution approaches the parabolic distribution of pipe flow. As $r_1 \to r_2$ this distribution approaches that of parallel-plate flow. These two conclusions are not obvious and are presented as Problem 7.48 at the end of this chapter.

Laminar Flow between Parallel Plates

For incompressible, steady, developed flow of a fluid between parallel plates, with the upper plate moving with velocity U.

Elemental Approach



Developed flow between parallel plates.

- For an elemental volume of unit depth (in the z-direction)
  - One dimensional flow, no acceleration, developed flow.

$$u(y) = \frac{1}{2\mu}\frac{d(p+\gamma h)}{dx}(y^2 - ay) + \frac{U}{a}y$$

Laminar Flow between Parallel Plates

Elemental Approach

$$u(y) = \frac{1}{2\mu}\frac{d(p + \gamma h)}{dx}(y^2 - ay) + \frac{U}{a}y$$

- **Couette Flow**: A flow with a **linear profile** resulting from the motion of the plate only.
- **Poiseuille Flow**: If the motion is only due to the pressure gradient (with U = 0).

Laminar Flow between Parallel Plates

Solving the Navier-Stokes Equations

- For a developed flow between parallel plates:
  - Streamlines are parallel to the plates so u = u(y); v = w = 0.

$$\rho\left(\underbrace{\cancel{\frac{\partial u}{\partial t}}}_{\text{steady}} + u\underbrace{\cancel{\frac{\partial u}{\partial x}}}_{\text{developed flow}} + \underbrace{\cancel{v}}_{0}\frac{\partial u}{\partial y} + \underbrace{\cancel{w}}_{0}\frac{\partial u}{\partial z}\right)$$

$$= -\frac{\partial p}{\partial x} + \mu\left(\underbrace{\cancel{\frac{\partial^2 u}{\partial x^2}}}_{\text{developed flow}} + \frac{\partial^2 u}{\partial y^2} + \underbrace{\cancel{\frac{\partial^2 u}{\partial z^2}}}_{\text{wide channel}}\right) + \gamma \sin\theta$$

$$u(y) = \frac{\lambda}{2}(y^2 - ay) + \frac{U}{a}y$$
$$= \frac{1}{2\mu}\frac{d(p+\gamma h)}{dx}(y^2 - ay) + \frac{U}{a}y$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{1}{\mu}\frac{d(p+\gamma h)}{dx}$$

This analysis applies to the midsection away from the side-walls

Double-integrating with u = 0, y = 0; u = U; y = a.

Laminar Flow between Parallel Plates

Simplified Flow Situation

- Velocity distribution between fixed plates (U = 0) is:

$$u(y) = \frac{1}{2\mu}\frac{d(p+\gamma h)}{dx}(y^2 - ay)$$

Flow rate per unit width:

$$Q = \int u\, dA$$

$$= \int_0^a \frac{1}{2\mu}\frac{d(p+\gamma h)}{dx}(y^2 - ay)\, dy = -\frac{a^3}{12\mu}\frac{d(p+\gamma h)}{dx}$$

Average velocity:

$$V = \frac{Q}{a \times 1}$$

$$= -\frac{a^2}{12\mu}\frac{d(p+\gamma h)}{dx}$$

Laminar Flow between Parallel Plates

Simplified Flow Situation

- Velocity distribution between fixed plates (U = 0) is: $u(y) = \frac{1}{2\mu}\frac{d(p+\gamma h)}{dx}(y^2 - ay)$

The pressure drop in terms of average velocity (horizontal channel): $\Delta p = \frac{12\mu VL}{a^2}$

- For plates on an incline, p is replaced with (p + γh)

The maximum velocity occurs at y = 0.5a and is: $u_{max} = -\frac{a^2}{8\mu}\frac{dp}{dx}$

Hence, average and maximum velocities are related by: $V = \frac{2}{3}u_{max}$

The pressure drop Δp over a length L of horizontal channel is:

$$\Delta p = \frac{2\tau_0}{a}L \qquad \tau_0 = -\frac{a}{2}\frac{dp}{dx}$$

Laminar Flow between Parallel Plates

Simplified Flow Situation

- **Friction factor, f**: Dimensionless wall shear valid for both laminar and turbulent flow.

$$f = \frac{\tau_0}{\frac{1}{8}\rho V^2}$$

The pressure drop in terms of friction factor is:

$$\frac{\Delta p}{\gamma} = f\frac{L}{2a}\frac{V^2}{2g}$$

$$f = \frac{8}{\rho V^2}\left(-\frac{a}{2}\frac{dp}{dx}\right) = \frac{8}{\rho V^2}\left(-\frac{a}{2}\right)\left(-\frac{12\mu V}{a^2}\right) = \frac{48\mu}{\rho a V} = \frac{48}{\mathrm{Re}}$$

- **Head Loss**: Pressure drop due to friction as fluid flows through a pipe.

$$\frac{\Delta p}{\gamma} = f\frac{L}{2a}\frac{V^2}{2g} \longrightarrow h_L = f\frac{L}{2a}\frac{V^2}{2g}$$

$$h_L = \frac{12\mu L V}{r a^2}$$

Using definitions for Reynolds number and friction factor; seen that head loss is directly proportional to average velocity.

Water at 20°C flows with a Reynolds number of 1500 between the 500-mm-wide, horizontal plates shown in Figure E7.3. Calculate

(a) the flow rate,

(b) the wall shear stress,

(c) the pressure drop over 3 m, and

(d) the velocity at $y = 0.5$ cm

Figure E7.3

Solution

Since the Reynolds number is 1500, the laminar flow equations are assumed applicable.

(a) Using the definition of the Reynolds number, the average velocity is found as follows:

$$1500 = \frac{Va}{\nu}$$

$$\therefore V = \frac{1500\nu}{a} = \frac{1500 \times 10^{-6}}{1.2 \times 10^{-2}} = 0.14 \text{ m/s}$$

Thus

$$Q = AV = 0.125 \times 0.12 \times 0.5 = \underline{7.5 \times 10^{-4} \text{ m}^3\text{/s}}$$

(b) Using Eq. 7.4.17, the pressure drop over length L is

$$\frac{\Delta p}{L} = \frac{12\mu V}{a^2} = \frac{12 \times 10^{-3} \times 0.125}{0.012^2} = 10.4 \text{ Pa/m}$$

The shearing stress at the wall is found, using Eq. 7.4.22, to be

$$\tau_0 = \frac{a}{2}\frac{\Delta p}{L} = \frac{0.012}{2} \times 10.4 = \underline{0.0624 \text{ Pa}}$$

(c) The pressure drop over 3 m is found to be

$$\Delta p = 10.4L = 10.4 \times 3 = \underline{31.2 \text{ Pa}}$$

Laminar Flow between Parallel Plates

(d) The velocity distribution of Eq. 7.4.14 is

$$u(y) = \frac{1}{2\mu}\frac{dp}{dx}(y^2 - ay)$$

$$= \frac{1}{2 \times 10^{-3}}(-10.4)\left(y^2 - 0.012y\right) = -5200(y^2 - 0.012y)$$

where we have used $dp/dx = -\Delta p/L$. At $y = 5$ mm, the velocity is

$$u = -5200\,(0.005^2 - 0.012 \times 0.005) = \underline{0.182 \text{ m/s}}$$

We have used three significant digits since the fluid properties are assumed known to three significant digits.

Laminar Flow between Parallel Plates

Find an expression for the pressure gradient between two parallel plates that results in a zero shear stress at the lower wall, where $y = 0$; also, sketch the velocity profiles for a top plate speed of U with various pressure gradients. Assume horizontal plates.

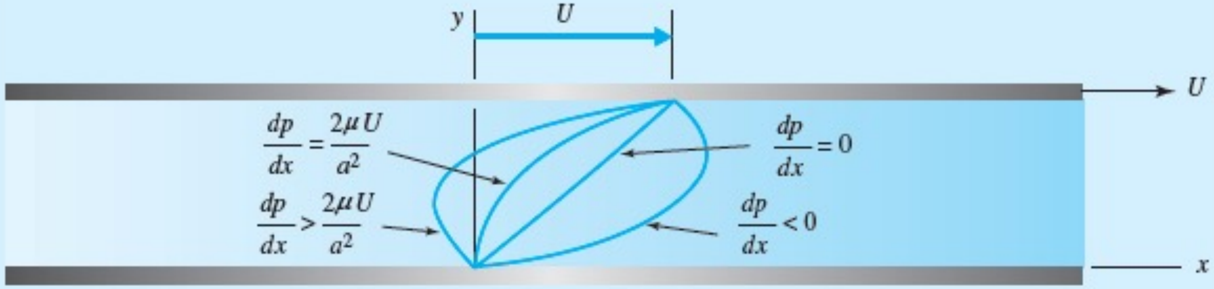


Figure E7.4

Solution

The velocity distribution for plates with the top plate moving with velocity U is given by Eq. 7.4.17. Letting $dh/dx = 0$, we have

$$u(y) = \frac{1}{2\mu}\frac{dp}{dx}(y^2 - ay) + \frac{U}{a}y$$

The shear stress is

$$\tau = \mu\frac{du}{dy}$$
$$= \frac{1}{2}\frac{dp}{dx}(2y - a) + \mu\frac{U}{a}$$

Laminar Flow between Parallel Plates

If $\tau = 0$ at $y = 0$, then $du/dy = 0$ at $y = 0$ and the pressure gradient is

$$\frac{dp}{dx} = \frac{2\mu U}{a^2}$$

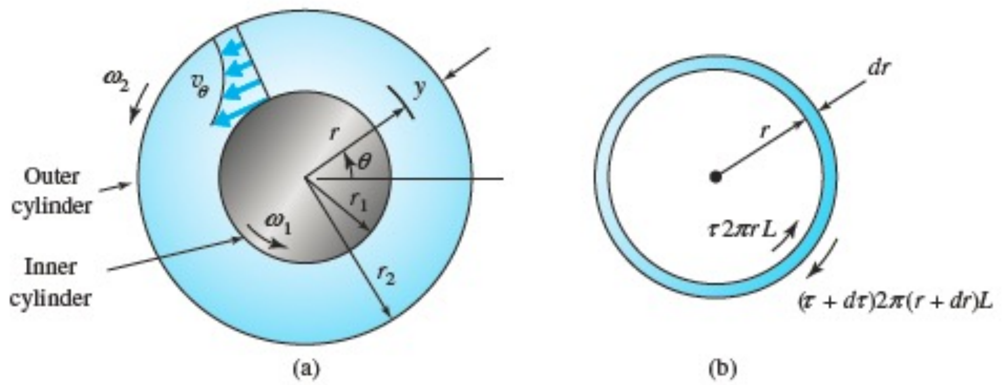
If dp/dx is greater than this value, the slope du/dy at $y = 0$ is negative and thus the velocity u will be negative near $y = 0$. If $dp/dx = 0$, we observe that a linear velocity distribution results, namely,

$$u(y) = \frac{U}{a}y$$

If dp/dx is negative, $u(y)$ is greater at each y-location than the linear distribution since $(y^2 - ay)$ is a negative quantity for all y's of interest.

All of the results above can be qualitatively displayed on a sketch of $u(y)$ for several dp/dx as shown in Figure E7.4.

Laminar Flow between Rotating Cylinders



Flow between concentric vertical cylinders: (a) basic flow variables; (b) element from between the cylinders.

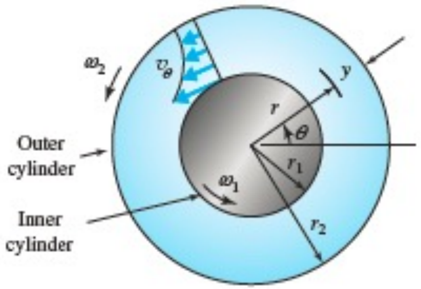
- Fully developed, steady flow between concentric, rotating cylinders.
  - Laminar flow is valid up to Re = 1700
  - Above this, a secondary laminar flow may develop, and eventually a turbulent flow develops.

Laminar Flow between Rotating Cylinders

Elemental Approach

- Neglecting body forces (vertical cylinder).
- Pressure doesn't vary with θ; resultant torque acting on an element (thin cylindrical shell) is zero as there is no angular acceleration.

$$\tau 2\pi r L \times r - (\tau + d\tau) 2\pi \,(r + dr) L \times (r + dr) = 0 \longrightarrow 2\tau + r\frac{d\tau}{dr} = 0$$



$$v_\theta(r) = \frac{A}{2}r + \frac{B}{r}$$

With the constants to be found by evaluating the boundary conditions:

$$v_\theta = r_1\omega_1 \text{ at } r = r_1, \text{ and } v_\theta = r_2\omega_2 \text{ at } r = r_2$$

$$A = 2\frac{\omega_2 r_2^2 - \omega_1 r_1^2}{r_2^2 - r_1^2} \qquad B = \frac{r_1^2 r_2^2(\omega_1 - \omega_2)}{r_2^2 - r_1^2}$$

Laminar Flow between Rotating Cylinders

Solving the Navier-Stokes Equations

- Steady, laminar flow between concentric cylinders.
  - Hence, circular streamlines $v_r = v_z = 0,\ v_\theta = v_\theta(r)$ only, and $\partial p/\partial\theta = 0$

$$\left(\underbrace{\cancel{\frac{\partial v_\theta}{\partial t}}}_{\text{steady}} + \underbrace{\cancel{v_r}}_{0}\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\underbrace{\cancel{\frac{\partial v_\theta}{\partial \theta}}}_{\text{symmetric flow}} + v_z\underbrace{\cancel{\frac{\partial v_\theta}{\partial z}}}_{0} + \underbrace{\cancel{\frac{v_r v_\theta}{r}}}_{0}\right)$$

$$= -\frac{1}{r}\underbrace{\cancel{\frac{\partial p}{\partial \theta}}}_{0} + \mu\left[\frac{\partial^2 v_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial v_\theta}{\partial r} + \frac{1}{r^2}\underbrace{\cancel{\frac{\partial^2 v_\theta}{\partial \theta^2}}}_{\text{symmetric flow}} + \underbrace{\cancel{\frac{\partial^2 v_\theta}{\partial z^2}}}_{\text{long cylinders}} + \frac{2}{r^2}\underbrace{\cancel{\frac{\partial v_r}{\partial \theta}}}_{\text{symmetric flow}} - \frac{v_\theta}{r^2}\right]$$

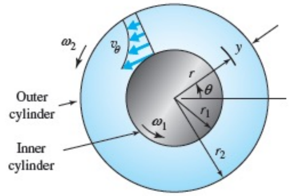
Cancelling the terms and double-integration leads to the same equations as in the previous slide.

$$v_\theta(r) = \frac{A}{2}r + \frac{B}{r}$$

$$A = 2\frac{r_2^2\omega_2 - r_1^2\omega_1}{r_2^2 - r_1^2} \qquad B = \frac{r_1^2 r_2^2(\omega_1 - \omega_2)}{r_2^2 - r_1^2}$$

Laminar Flow between Rotating Cylinders

Flow with the Outer Cylinder Fixed ($\omega_2 = 0$)



- E.g., For a shaft rotating in a bearing.

Velocity Distribution: $$v_\theta = \frac{r_1^2\omega_1}{r_2^2 - r_1^2}\left(\frac{r_2^2}{r} - r\right)$$

Shearing stress: $$\tau_1 = -\left[\mu r \frac{d}{dr}\left(\frac{v_\theta}{r}\right)\right]_{r=r_1} = \mu \frac{2}{r_1^2}\frac{r_1^2 r_2^2 \omega_1}{r_2^2 - r_1^2} = \frac{2\mu r_2^2\omega_1}{r_2^2 - r_1^2}$$

Torque, T to rotate the inner cylinder of length, L:

$$T = \tau_1 A_1 r_1$$

$$T = \frac{2\mu r_2^2 \omega_1}{r_2^2 - r_1^2} 2\pi r_1 L r_1 = \frac{4\pi\mu r_1^2 r_2^2 L\omega_1}{r_2^2 - r_1^2}$$

Power to rotate the shaft (multiply torque by rotational speed):

$$\dot{W} = T\omega_1 = \frac{4\pi\mu r_1^2 r_2^2 L\omega_1^2}{r_2^2 - r_1^2}$$

Power needed to overcome resistance of viscosity → Leads to an increase in internal energy and temperature of the fluid.

Laminar Flow between Rotating Cylinders

Estimate the viscosity of an oil contained in the annulus between two 25-cm-long cylinders, as shown in Figure E7.6. The outer stationary cylinder is 80 mm in diameter. The 78-mm-diameter inner cylinder rotates at 3800 rpm when a torque of 1.2 N · m is applied. The specific gravity of the oil is 0.85. Neglect any torque due to the cylinder ends.

Solution

Assuming that the Reynolds number is less than 1700, Eq. 7.5.19 provides

$$\mu = \frac{T\left(r_2^2 - r_1^2\right)}{4\pi r_1^2 r_2^2 L\omega_1}$$

$$= \frac{1.2\ \text{N}\cdot\text{m}(0.04^2 - 0.039^2)\ \text{m}^2}{4\pi \times 0.04^2\ \text{m}^2 \times 0.039^2\ \text{m}^2 \times 0.25\ \text{m} \times (3800 \times 2\pi/60)\ \text{rad/s}} = \underline{0.0312\ \text{N}\cdot\text{s/m}^2}$$

Check the Reynolds number using $\nu = \mu/\rho$:

$$\text{Re} = \frac{\omega_1 r_1 \delta}{\nu} = \frac{(3800 \times 2\pi/60)\ \text{rad/s} \times 0.039\ \text{m} \times 0.002/2\ \text{m}}{0.0312/(1000 \times 0.85)\ \text{m}^2/\text{s}} = 423$$

This is less than 1700 so the calculation is acceptable.

Laminar Flow between Rotating Cylinders

Show that as the inner cylinder radius of Figure E7.6 approaches the outer cylinder radius the velocity distribution approaches the linear distribution between parallel plates with one plate moving and a zero pressure gradient. This is Couette flow.

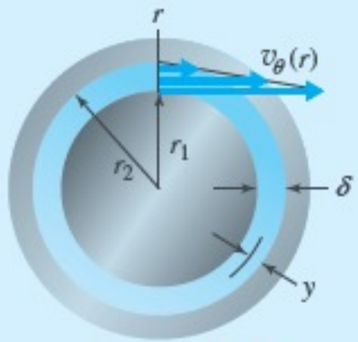


Figure E7.6

Solution

For this problem we will let $\omega_2 = 0$; the velocity distribution (7.5.17) is

$$v_\theta(r) = \frac{r_1^2\omega_1}{r_2^2 - r_1^2}\left(\frac{r_2^2}{r} - r\right)$$

$$= \frac{r_1^2\omega_1}{r_2^2 - r_1^2}\frac{r_2^2 - r^2}{r} = \frac{r_1^2\omega_1}{r_2^2 - r_1^2}(r_2 - r)\frac{r_2 + r}{r}$$

Laminar Flow between Rotating Cylinders

Introduce the independent variable y, measured from the outer cylinder defined by $r + y = r_2$ (see Figure 7.6); let $\delta = r_2 - r_1$. Then the above can be written as

$$v_\theta(r) = \frac{r_1^2\omega_1(r_2 - r)}{(r_2 - r_1)(r_2 + r_1)}\frac{r_2 + r}{r}$$
$$= \frac{r_1^2\omega_1 y}{\delta(r_2 + r_1)}\frac{2r_2 - y}{r_2 - y}$$

As the inner radius approaches the outer radius we can write $r_1 \simeq r_2$. Letting $r_2 \simeq r_1 \simeq R$ we have $r_2 + r_1 \simeq 2R$ and

$$\frac{2R - y}{R - y} \simeq 2$$

since $y \ll R$. The velocity distribution then simplifies to

$$v_\theta(y) = \frac{R^2\omega_1 y}{\delta 2R} \times 2 = \frac{R\omega_1}{\delta} y$$

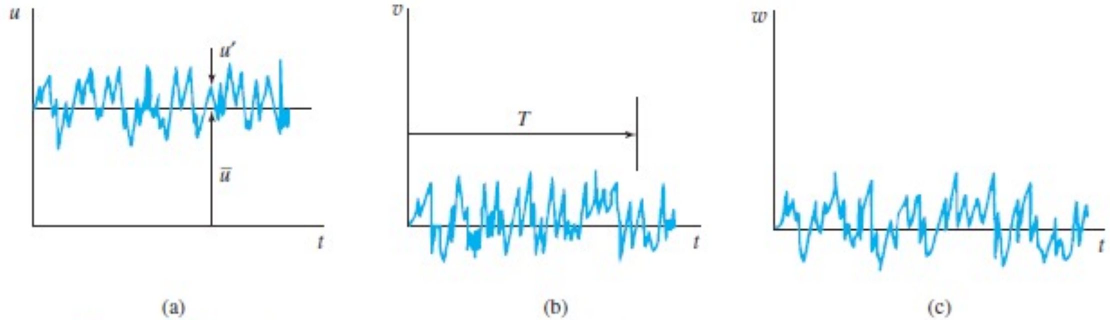
This is a linear distribution and is a good approximation to the flow whenever $\delta \ll R$.

Turbulent Flow in a Pipe

- Developed turbulent flow in a circular pipe is of interest in practical applications (most flows in pipes are turbulent).
  - Laminar flows have been seen in Reynolds numbers of 40,000 in a pipe flow.
  - In standard conditions, $Re_{turb} = 2000$.

- All three velocity components are nonzero.
  - Need time-average quantities.

$$\overline{u} = \frac{1}{T}\int_0^T u(t)\,dt$$



Velocity components in a steady turbulent pipe flow: (a) x-component velocity u; (b) r-component velocity v; (c) θ-component velocity w.

Turbulent Flow in a Pipe

Show that $\overline{u'} = 0$ and $\overline{\dfrac{\partial u}{\partial y}} = \dfrac{\partial \overline{u}}{\partial y}$ for a turbulent flow.

Solution

To show that $\overline{u'} = 0$ we simply substitute the expression (7.6.1) for $u(t)$ into Eq. 7.6.2 and obtain

$$\begin{aligned}\overline{u} &= \frac{1}{T}\int_0^T (\overline{u} + u')\,dt \\ &= \frac{1}{T}\int_0^T \overline{u}\;dt + \frac{1}{T}\int_0^T u'\,dt \\ &= \overline{u}\frac{1}{T}\int_0^T dt + \overline{u'} \\ &= \overline{u} + \overline{u'}\end{aligned}$$

Subtracting $\overline{u}$ from both sides results in

$$\overline{u'} = 0$$

Now, let us time average the derivative $\partial u/\partial y$. We have

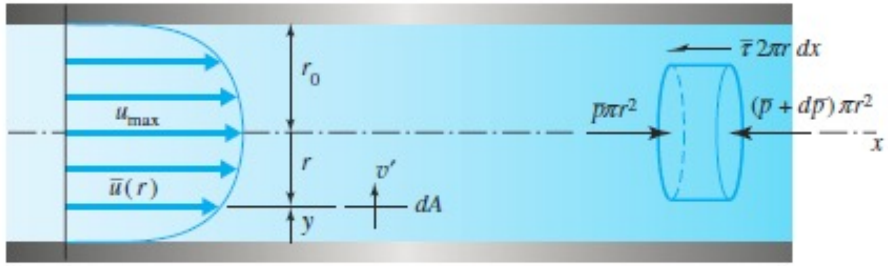
$$\begin{aligned}\overline{\frac{\partial u}{\partial y}} &= \frac{1}{T}\int_0^T \frac{\partial u}{\partial y}\,dt \\ &= \frac{\partial}{\partial y}\left(\frac{1}{T}\int_0^T u\,dt\right) = \frac{\partial}{\partial y}\overline{u}\end{aligned}$$

since T is a constant. Thus

$$\overline{\frac{\partial u}{\partial y}} = \frac{\partial \overline{u}}{\partial y}$$

Turbulent Flow in a Pipe

Differential Equation



Turbulent flow in a horizontal pipe.

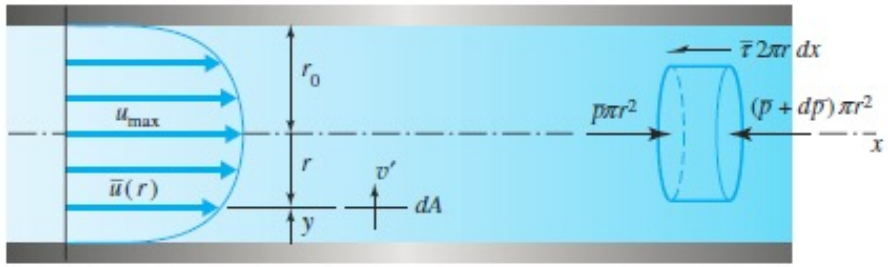
- The differential x-force from the random motion of a fluid particle through an incremental area dA is: $$dF = -\rho v'\,dA\,u'$$

- u' is the negative change in x-component velocity.

- The *turbulent shear stress* is: $$\tau_{turb} = \frac{dF}{dA} = -\rho u'v'$$
- Time-average turbulent shear stress is the *apparent* shear stress. $$\bar{\tau}_{turb} = -\rho\overline{u'v'}$$

Turbulent Flow in a Pipe

Differential Equation



Turbulent flow in a horizontal pipe.

- The total shear stress (from viscosity and momentum exchange) due to laminar and turbulent effects would be:

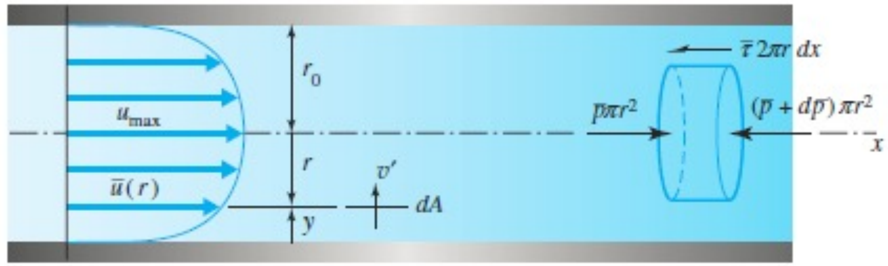
$$\bar{\tau} = \bar{\tau}_{\text{lam}} + \bar{\tau}_{\text{turb}}$$
$$= \mu\frac{\partial \bar{u}}{\partial y} - \rho\overline{u'v'}$$

- The shear stress related to the pressure gradient is:

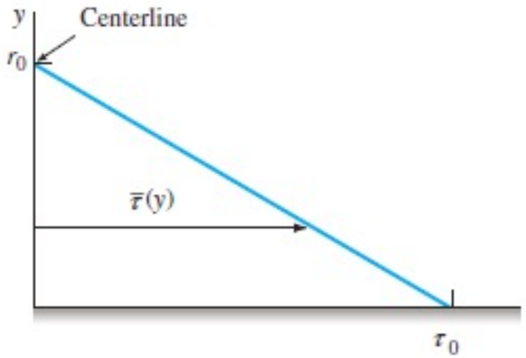
$$\bar{\tau} = -\frac{r}{2}\frac{d\bar{p}}{dx} = \frac{r\Delta\bar{p}}{2L}$$

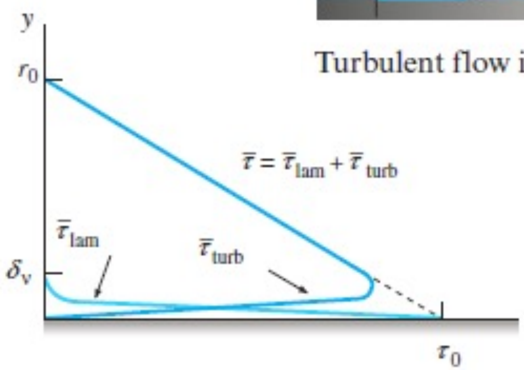
Turbulent Flow in a Pipe

Differential Equation



Turbulent flow in a horizontal pipe.

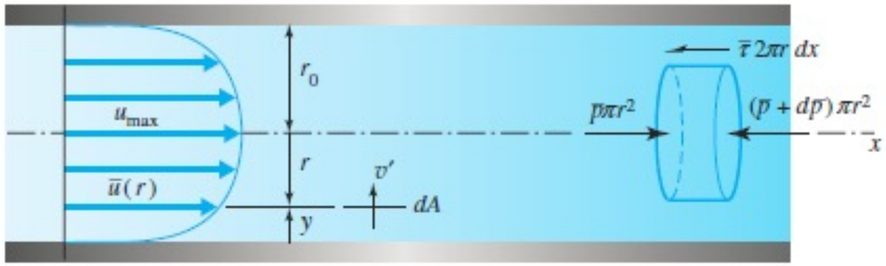




- Shear stress distribution is linear for turbulent and laminar flow.
- Turbulent shear goes to zero at the wall.
- Total shear at the centerline is zero.

Turbulent Flow in a Pipe

Differential Equation



Turbulent flow in a horizontal pipe.

- To find the time-average velocity distribution, the differential equation is formed from equations on the previous slide.

$$\frac{r}{2}\frac{d\overline{p}}{dx} = \rho\overline{u'v'} + \mu\frac{d\overline{u}}{dr}$$

The term $\overline{u'v'}$ cannot be determined analytically.

- Have Eddy viscosity: *(Parameterization of Eddy Momentum Flux, Reynolds stresses)*

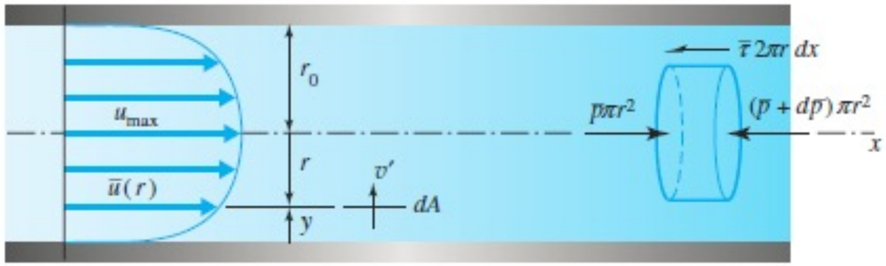
$$\overline{u'v'} = \eta\frac{d\overline{u}}{dy}$$

- Hence:

$$\frac{r}{2}\frac{d\overline{p}}{dx} = \rho(\nu + \eta)\frac{d\overline{u}}{dr}$$

Turbulent Flow in a Pipe

Differential Equation



Turbulent flow in a horizontal pipe.

- For turbulent flow, it is helpful to define a **mixing length l_m**:
  - Distance a particle moves before interacting with another particle.

$$\eta = l_m^2 \left| \frac{d\bar{u}}{dy} \right|$$

- The **correlation coefficient K_{uv}** is a normalized turbulent shear stress.
  - Has limits of ±1
  - With time-averaged quantities.

$$K_{uv} = \frac{\overline{u'v'}}{\sqrt{\overline{u'^2}}\sqrt{\overline{v'^2}}}$$

Turbulent Flow in a Pipe

Note that in Figure 7.9b there is a region near the wall where the turbulent shear is near its maximum and is relatively constant, as shown in the expanded view of Figure E7.8, and the viscous shear is quite small. Assume that the mixing length is directly proportional to the distance from the wall. With this assumption show that $\bar{u}(y)$ is logarithmic in this region near the wall.

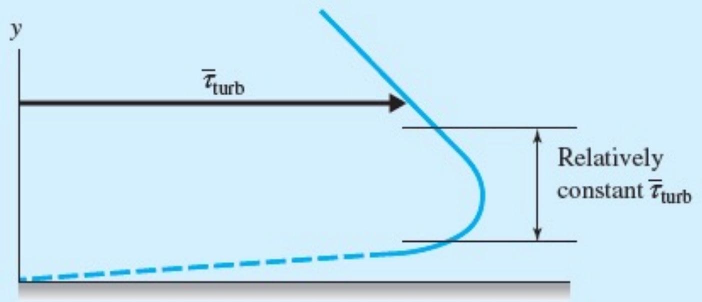


Figure E7.8

Solution

If the viscous shear is negligible (as it is away from the thin wall layer), we have, combining Eqs. 7.6.5 and 7.6.8, and 7.6.10,

$$\bar{\tau}_{turb} = \rho\eta \frac{d\bar{u}}{dy} = \rho l_m^2 \left(\frac{d\bar{u}}{dy} \right)^2$$

Turbulent Flow in a Pipe

Now, if $\overline{\tau}_{\text{turb}} = \text{const.} = c_1$ and we assume, as given in the problem statement, that

$$l_m = c_2 y$$

there results

$$c_1 = \rho c_2^2 y^2 \left(\frac{du}{dy}\right)^2$$

or

$$y\frac{d\overline{u}}{dy} = c_3$$

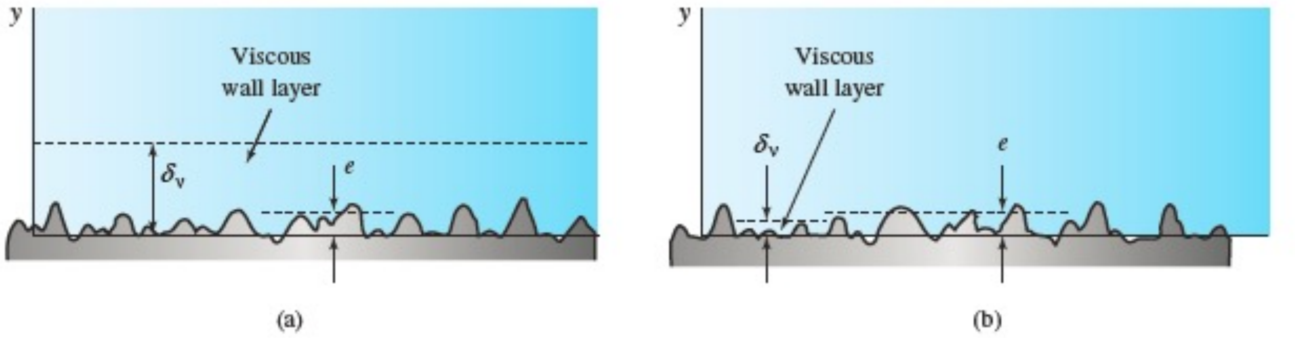
where $c_3 = \sqrt{c_1/\rho c_2^2}$. This is integrated to yield

$$\overline{u}(y) = c_3 \ln y + c_4$$

Hence, with the foregoing assumptions we see that a logarithmic profile is predicted for the region of constant turbulent shear near the wall. This is, in fact, observed from experimental data; so we conclude that the above assumptions are reasonable for a turbulent flow in a pipe.

Turbulent Flow in a Pipe

Velocity Profile



(a) A smooth wall and (b) a rough wall.

e = Average wall roughness height

δ_v = Viscous wall layer thickness

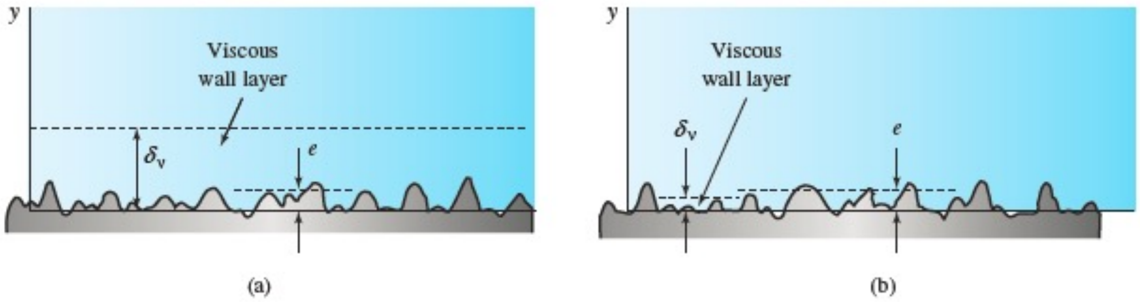
- *Hydraulically smooth*: The viscous wall thickness (δ_v) is large enough that it submerges the wall roughness elements→ Negligible effect on the flow (almost as if the wall is smooth).

- If the viscous wall layer is very thin→ Roughness elements protrude off the layer→ The wall is rough.

- The relative roughness e/D and Reynolds number can be used to find if a pipe is smooth/rough.

Turbulent Flow in a Pipe

Velocity Profile



(a) A smooth wall and (b) a rough wall.

e = Average wall roughness height

δ_v = Viscous wall layer thickness

- For a smooth wall, there are two regions of flow (wall and outer regions).
- Wall region: Characteristic velocity = **shear velocity** $u_\tau = \sqrt{\frac{\tau_0}{\rho}}$;

  Characteristic length = **viscous length** $\frac{\nu}{u_\tau}$

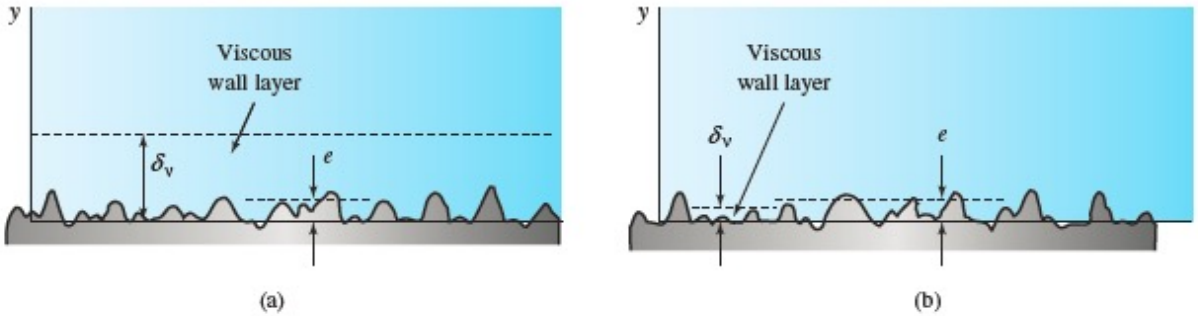
$$\frac{\bar{u}}{u_\tau} = \frac{u_\tau y}{\nu} \text{ (viscous wall layer)} \qquad 0 \leq \frac{u_\tau y}{\nu} \lesssim 5$$

$$\frac{\bar{u}}{u_\tau} = 2.44 \ln \frac{u_\tau y}{\nu} + 4.9 \text{ (turbulent region)} \qquad 30 < \frac{u_\tau y}{\nu}, \ \frac{y}{r_0} < 0.15$$

Dimensionless velocity distribution in the wall region for a smooth pipe

Turbulent Flow in a Pipe

Velocity Profile



(a) A smooth wall and (b) a rough wall.

e = Average wall roughness height

δ_v = Viscous wall layer thickness

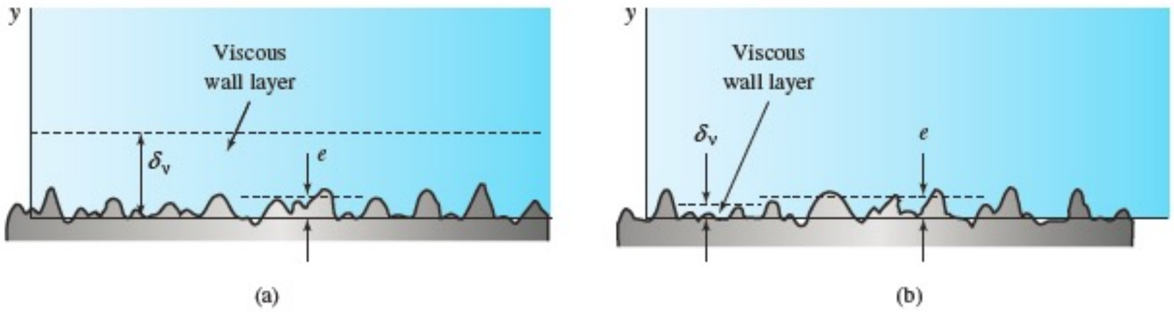
- For rough pipes, the viscous wall layer doesn't play an important role.
    - Turbulence starts from the protruding wall elements.
- Characteristic length is the **Average roughness height** *e*

$$\frac{\bar{u}}{u_\tau} = 2.44 \ln\frac{y}{e} + 8.5 \qquad \frac{y}{r_0} < 0.15$$

Dimensionless velocity profile for the wall region of a rough pipe

Turbulent Flow in a Pipe

Velocity Profile



(a) A smooth wall and (b) a rough wall.

e = Average wall roughness height

δ_v = Viscous wall layer thickness

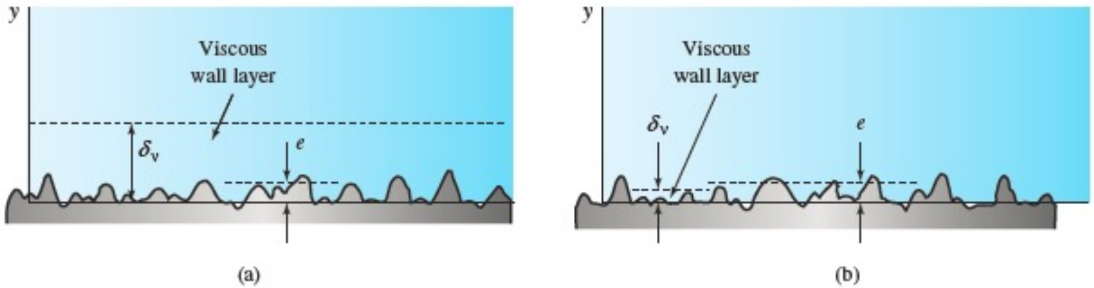
- In the outer region, **characteristic length is r_0**

$$\frac{u_{\max} - \bar{u}}{u_\tau} = 2.44 \ln \frac{r_0}{y} + 0.8 \quad \frac{y}{r_0} \leq 0.15 \qquad \text{(outer region)}$$

Velocity defect $(u_{max} - \bar{u})$ is normalized with u_τ. Relation is for both smooth and rough pipes.

Turbulent Flow in a Pipe

Velocity Profile



(a) A smooth wall and (b) a rough wall.

e = Average wall roughness height

δ_v = Viscous wall layer thickness

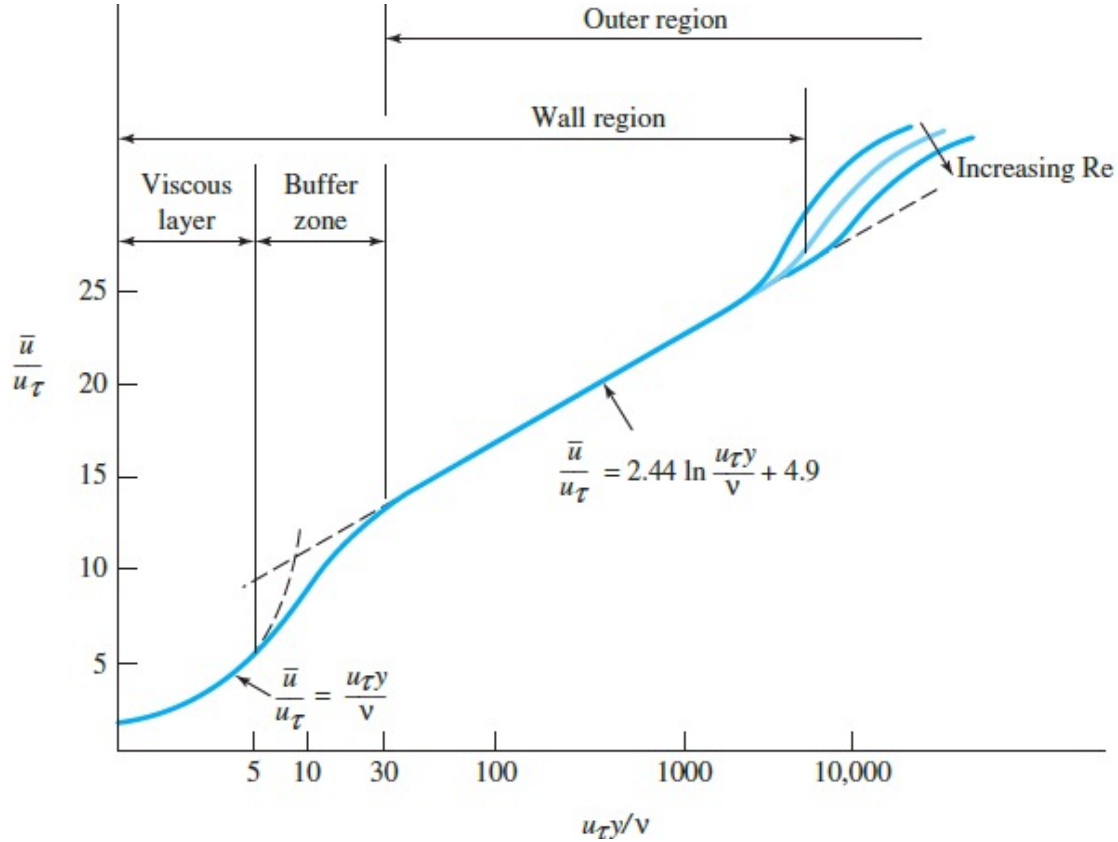
- The wall and outer regions may overlap. The maximum velocity is:

$$\frac{u_{max}}{u_\tau} = 2.44 \ln \frac{u_\tau r_0}{\nu} + 5.7 \qquad \text{(smooth pipes)}$$

$$\frac{u_{max}}{u_\tau} = 2.44 \ln \frac{r_0}{e} + 9.3 \qquad \text{(rough pipes)}$$

Turbulent Flow in a Pipe

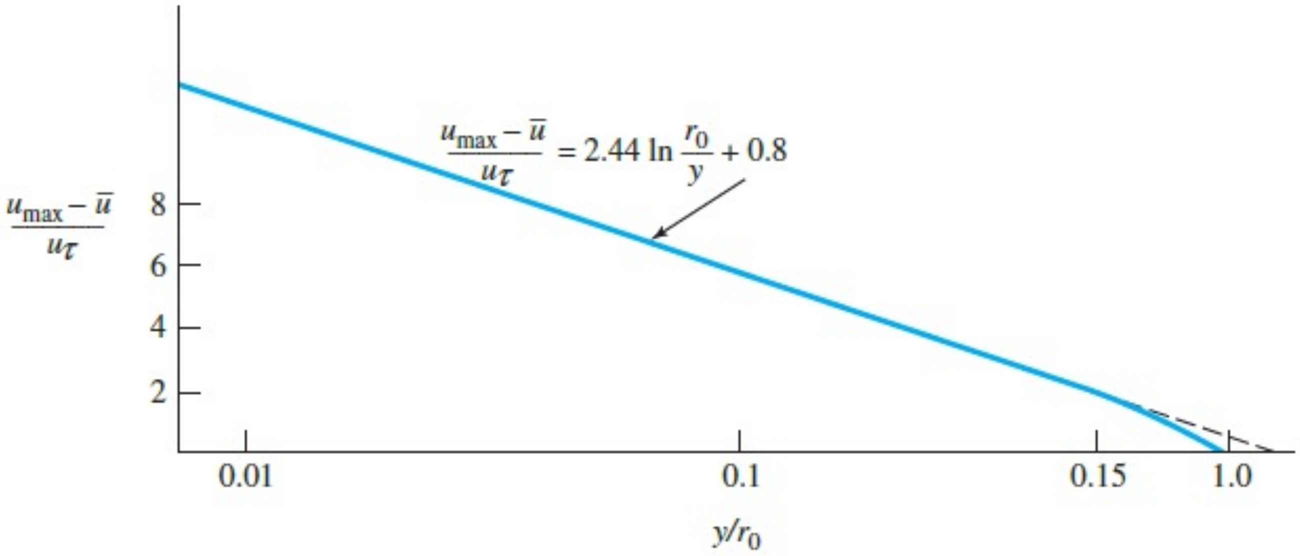
Velocity Profile



Wall Region (Empirical relations for turbulent flow in a smooth pipe)

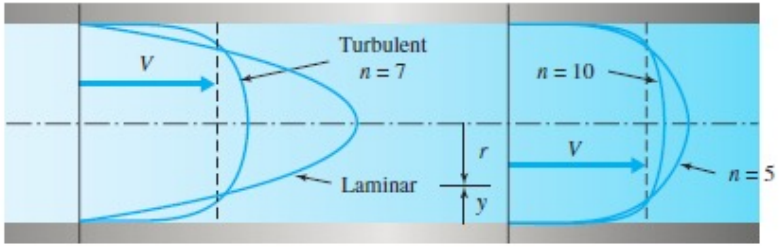
Turbulent Flow in a Pipe

Velocity Profile



Outer Region (Empirical relations
for turbulent flow in a smooth pipe)

Turbulent Flow in a Pipe



Turbulent velocity profile.

- The *power-law profile* describes the turbulent flow velocity distribution in a pipe:
  - Simpler form

$$\frac{\bar{u}}{u_{max}} = \left(\frac{y}{r_0}\right)^{1/n}$$

- Average velocity is then calculated to be:

$$V = \frac{1}{\pi r_0^2}\int_0^{r_0} \bar{u}(r) 2\pi r\ dr = \frac{2n^2}{(n+1)(2n+1)} u_{max}$$

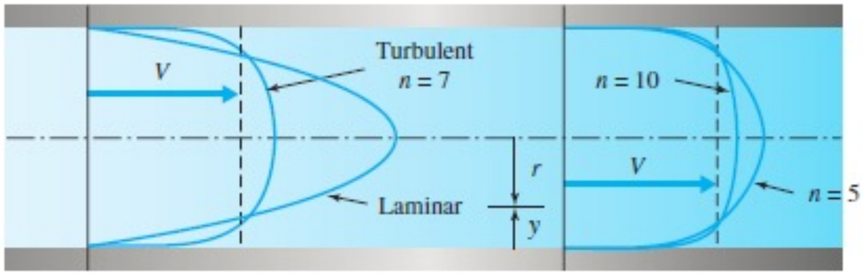
- n: Depends on the friction factor f (Reynolds number and pipe wall roughness)

$$n = \frac{1}{\sqrt{f}}$$

Table 7.1 Exponent n for Smooth Pipes

$Re = VD/\nu$	4×10^3	10^5	10^6	$> 2 \times 10^6$
n	6	7	9	10

Turbulent Flow in a Pipe



Turbulent velocity profile.

- The *power-law profile* for turbulent velocity flow distribution:
  - Cannot be used to obtain the slope at the wall (infinite for all n).
    - Cannot be used to predict wall shear stress.

Turbulent Flow in a Pipe

Water at 20°C flows in a 100-mm-diameter pipe at an average velocity of 1.6 m/s. If the roughness elements are 0.046 mm high, would the wall be rough or smooth? Refer to Figure 7.10.

Solution

To determine if the wall is rough or smooth, we must compare the viscous wall layer thickness with the height of the roughness elements. So, let's find the viscous wall layer thickness. From Figure 7.11 the viscous layer thickness is determined by letting $u_\tau y/\nu = 5$, where $y = \delta_\nu$. First, we must find u_τ. The Reynolds number is

$$\begin{aligned} \text{Re} &= \frac{VD}{\nu} \\ &= \frac{1.6 \times 0.1}{10^{-6}} = 1.6 \times 10^5 \end{aligned}$$

From Table 7.1 $n \simeq 7.5$, so that, from Eq. 7.6.21,

$$\begin{aligned} f &= \frac{1}{n^2} \\ &= \frac{1}{7.5^2} = 0.018 \end{aligned}$$

Turbulent Flow in a Pipe

The wall shear is calculated from Eq. 7.3.19:

$$\tau_0 = \frac{1}{8}\rho V^2 f$$
$$= \frac{1}{8} \times 1000 \times 1.6^2 \times 0.018 = 5.8 \text{ Pa}$$

The friction velocity is found from the definition of the shear velocity:

$$u_\tau = \sqrt{\frac{\tau_0}{\rho}}$$
$$= \sqrt{\frac{5.8}{1000}} = 0.076 \text{ m/s}$$

This allows us to calculate the viscous wall layer thickness using $y = \delta_v$:

$$\delta_v = \frac{5\nu}{u_\tau}$$
$$= \frac{5 \times 10^{-6}\text{ m}^2\text{/s}}{0.076\text{ m/s}} = 6.6 \times 10^{-5}\text{ m} \qquad \text{or} \qquad 0.066\text{ mm}$$

Since the roughness elements are only 0.046 mm high, they are submerged in the viscous wall layer. Consequently, *the wall is smooth* (see Figure 7.10a). If the pipe were made of cast iron with $e = 0.26$ mm, the wall would be rough.

Note that the viscous wall layer, even at this relatively low velocity, is about 0.1% of the radius. The viscous wall layer is usually extremely thin.

Turbulent Flow in a Pipe

The 40-mm-diameter smooth, horizontal pipe of Figure E7.10 transports 0.004 m^3/s of water at 20°C. Using the power-law profile, approximate (a) the friction factor, (b) the maximum velocity, (c) the radial position where $u = V$, (d) the wall shear, (e) the pressure drop over a 10-m length, and (f) the maximum velocity using Eq. 7.6.16.

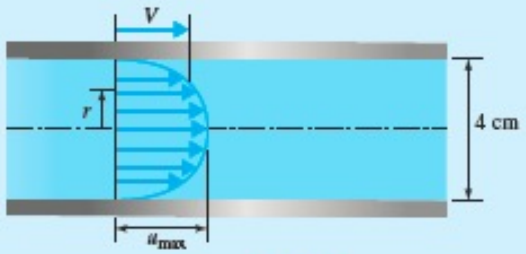


Figure E7.10

Solution

(a) The average velocity is calculated to be

$$V = \frac{Q}{A} = \frac{0.004}{\pi \times 0.02^2} = 3.18 \text{ m/s}$$

The Reynolds number is

$$\text{Re} = \frac{VD}{\nu} = \frac{3.18 \times 0.04}{10^{-6}} = 1.27 \times 10^5$$

From Table 7.1 we see that $n \cong 7.5$ and from Eq. 7.6.21,

$$f = \frac{1}{n^2} = \frac{1}{7.5^2} = \underline{0.018}$$

Turbulent Flow in a Pipe

(b) The maximum velocity is found using Eq. 7.6.20 to be

$$u_{max} = \frac{(n+1)(2n+1)}{2n^2}V$$
$$= \frac{8.5 \times 16}{2 \times 7.5^2} \times 3.18 = \underline{3.84 \text{ m/s}}$$

(c) The distance from the wall where $u = V = 3.18$ m/s is found using Eq. 7.6.19 as follows:

$$\frac{\overline{u}}{u_{max}} = \left(\frac{y}{r_0}\right)^{1/7.5}$$
$$\therefore y = r_0\left(\frac{u}{u_{max}}\right)^{7.5}$$
$$= 2\left(\frac{3.18}{3.84}\right)^{7.5} = 4.9 \text{ mm}$$

The radial position is thus

$$r = r_0 - y$$
$$= 20 - 4.9 = \underline{15.1 \text{ mm}}$$

(d) The wall shear is found using Eq. 7.3.19 and is

$$\tau_0 = \frac{1}{8}\rho V^2 f$$
$$= \frac{1}{8} \times 1000 \times 3.18^2 \times 0.018 = \underline{23 \text{ Pa}}$$

Turbulent Flow in a Pipe

(e) The pressure drop is calculated using Eq. 7.6.18 with $\Delta p/L = -dp/dx$ to be

$$\Delta p = \frac{2\tau_0 L}{r_0}$$

$$= \frac{2 \times 23 \times 10}{0.02} = 23\,000\ \text{Pa} \quad \text{or} \quad \underline{23\ \text{kPa}}$$

(f) To use Eq. 7.6.16 we must know the shear velocity. It is

$$u_\tau = \sqrt{\frac{\tau_0}{\rho}}$$

$$= \sqrt{\frac{23}{1000}} = 0.152\ \text{m/s}$$

We then find u_{max} to be, using $\nu = 10^{-6}\ \text{m}^2/\text{s}$,

$$u_{max} = 0.152\left(2.44\ \ln\frac{0.152 \times 0.02}{10^{-6}} + 5.7\right) = \underline{3.84\ \text{m/s}}$$

This is the same as that given by the power-law formula in part (b). This answer is considered to be more accurate if it differs from that of Eq. 7.6.20. Note that the experimental data do not allow for accuracy in excess of three significant digits, and often to only two significant digits.

Turbulent Flow in a Pipe

7.6.3 Losses in Developed Pipe Flow

- Most calculated quantity in pipe flow is the **head loss***.*
  - Allows pressure change to be found→ pump selection.
  - Derived from energy equation.

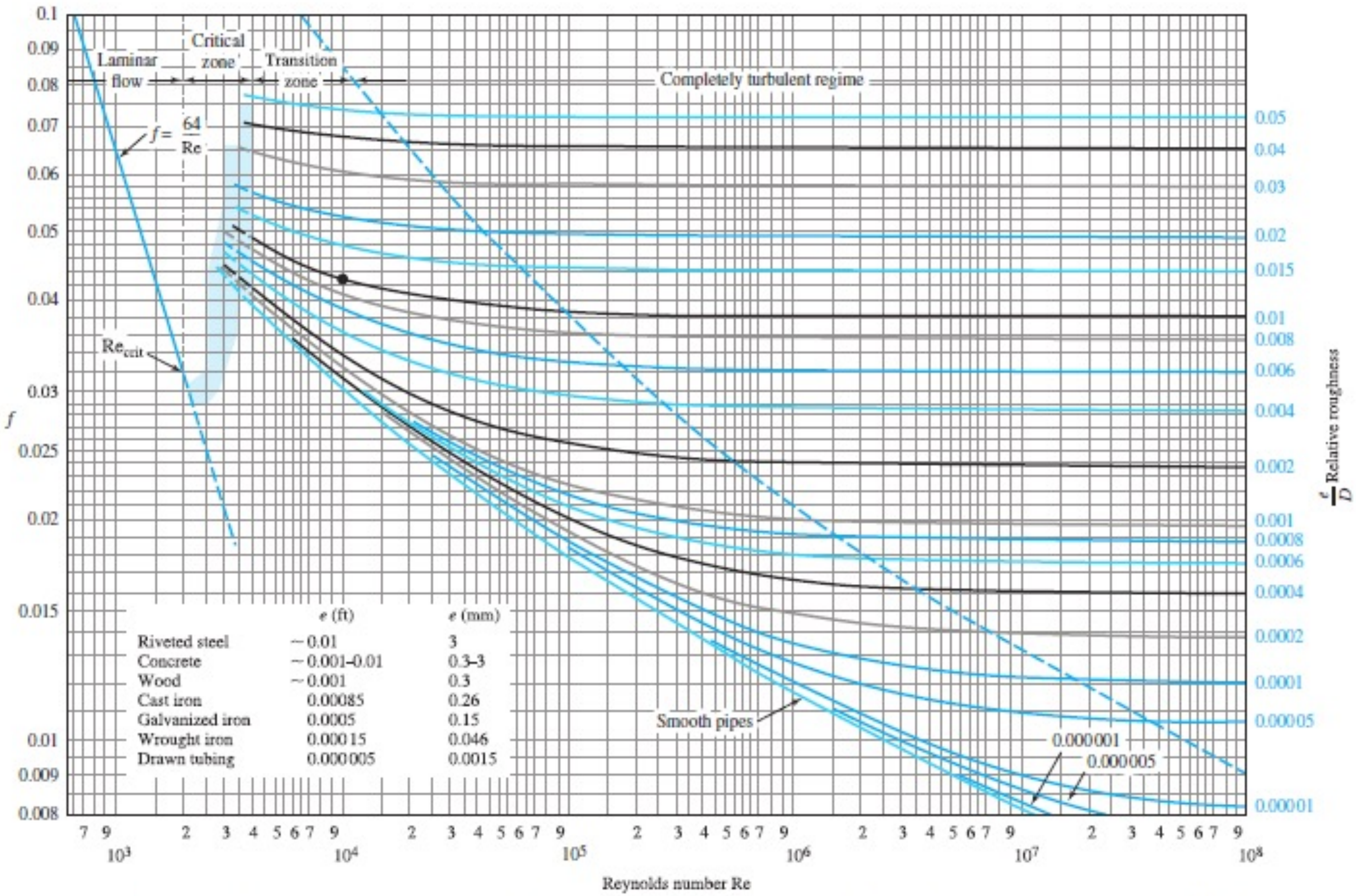
$$h_L = \frac{\Delta(p + \gamma h)}{\gamma}$$

$$h_L = f\frac{L}{D}\frac{V^2}{2g}$$

Head loss from wall shear in a developed flow is related to the friction factor(f).

- f = *f*(ρ, μ, V, D, e)
- **Darcy-Weisbach equation**

Turbulent Flow in a Pipe



Moody diagram. (From L. F. Moody, Trans. ASME, Vol. 66, 1944. Reproduced with permission of ASME.)
(Note: If $e/D = 0.01$ and $Re = 10^4$, the dot locates $f = 0.043$.)

Turbulent Flow in a Pipe

Losses in Developed Pipe Flow

- Moody diagram is a plot of experimental data relating friction factor to the Reynolds number.
  - For fully developed pipe flow over a range of wall roughnesses.

- For a given wall roughness→ There is a large enough Re to get a constant friction factor→ **Completely turbulent regime.**
- For smaller relative roughness→ As Re decreases, friction factor increases→ **Transition zone**→ Friction factor becomes like that of a smooth pipe.
- For Re < 2000→ The **critical zone** couples the turbulent flow to the laminar flow and may represent an oscillatory flow that alternately exists between turbulent and laminar flow.
- Assume new pipes→ As a pipe gets older, corrosion occurs changing both the roughness and the pipe diameter.

Turbulent Flow in a Pipe

Losses in Developed Pipe Flow

Smooth pipe flow: $$\frac{1}{\sqrt{f}} = 0.86 \ln \mathrm{Re}\sqrt{f} - 0.8$$

Empirical equations for Re > 4000

Completely turbulent zone: $$\frac{1}{\sqrt{f}} = -0.86 \ln \frac{e}{3.7D}$$

Transition zone: $$\frac{1}{\sqrt{f}} = -0.86 \ln\left(\frac{e}{3.7D} + \frac{2.51}{\mathrm{Re}\sqrt{f}}\right)$$

- **Colebrook equation**: The equation that couples the smooth pipe equation to the completely turbulent regime equation.
- Smooth pipe flow: $e = 0$; Completely turbulent zone: $\mathrm{Re} = \infty$

Turbulent Flow in a Pipe

Losses in Developed Pipe Flow

Category	*Known*	*Unknown*
1	Q, D, e, ν	h_L
2	D, e, ν, h_L	Q
3	Q, e, ν, h_L	D

- **Three types of problems for developed turbulent flow in a pipe:**
    - One: Straightforward. Needs no iteration when using the Moody diagram.
    - Two and Three: Engineering design situation. Needs an iterative trial-and-error process.

Turbulent Flow in a Pipe

Losses in Developed Pipe Flow

- **To avoid trial-and-error and the Moody diagram, use:**

$$h_L = 1.07\frac{Q^2L}{gD^5}\left\{\ln\left[\frac{e}{3.7D} + 4.62\left(\frac{\nu D}{Q}\right)^{0.9}\right]\right\}^{-2} \qquad \begin{matrix} 10^{-6} < e/D < 10^{-2} \\ 3000 < \mathrm{Re} < 3\times10^8 \end{matrix}$$

$$Q = -0.965\left(\frac{gD^5h_L}{L}\right)^{0.5}\ln\left[\frac{e}{3.7D} + \left(\frac{3.17\nu^2L}{gD^3h_L}\right)^{0.5}\right] \qquad \mathrm{Re} > 2000$$

$$D = 0.66\left[e^{1.25}\left(\frac{LQ^2}{gh_L}\right)^{4.75} + \nu Q^{9.4}\left(\frac{L}{gh_L}\right)^{5.2}\right]^{0.04} \qquad \begin{matrix} 10^{-6} < e/D < 10^{-2} \\ 5000 < \mathrm{Re} < 3\times10^8 \end{matrix}$$

- Developed by Swamee and Jain (1976) for pipe flow.
  - First and last equations are accurate to within 2% of the Moody diagram. The middle equation is as accurate as the Moody diagram.

Turbulent Flow in a Pipe

Water at 20°C is transported for 450 m in a 38-mm-diameter wrought iron horizontal pipe with a flow rate of 2.75 L/s. Calculate the head loss and the pressure drop over the 450 m length of pipe, using (a) the Moody diagram and (b) the alternate method.

Solution

(a) The average velocity is

$$V = \frac{Q}{A} = \frac{2.75 \times 10^{-3}}{\pi \times (0.019)^2} = 2.43\,\text{m/s}$$

The Reynolds number is

$$\text{Re} = \frac{VD}{\nu} = \frac{2.43 \times 0.038}{10^{-6}} = 92\,300$$

Obtaining e from Figure 7.13, we have, using $D = 0.038$ m,

$$\frac{e}{D} = \frac{0.000046}{0.038} = 0.0012$$

The friction factor is read from the Moody diagram to be

$$f = 0.023$$

The head loss is calculated as

$$h_L = f\frac{L}{D}\frac{V^2}{2g}$$

$$= 0.023\frac{450}{0.038}\frac{(2.43)^2}{2 \times 9.8} = \underline{82\text{ m}}$$

Turbulent Flow in a Pipe

This answer is given to two significant numbers since the friction factor is known to at most two significant numbers. The pressure drop is found by Eq. 7.6.22 to be

$$\begin{aligned}\Delta p &= \gamma h_L \\ &= 9810\ \mathrm{N/m^3} \times 82 = 804420\ \mathrm{N/m^2} \quad \text{or} \quad \underline{804.4\ \mathrm{kPa}}\end{aligned}$$

(b) The alternate method for this Category 1 problem uses Eq. 7.6.29, with $D = 0.038$ m:

$$\begin{aligned}h_L &= 1.07\frac{0.00275^2 \times 450}{9.81 \times 0.038^5}\left\{\ln\left[\frac{0.0012}{3.7} + 4.62\left(\frac{10^{-6} \times 0.038}{0.00275}\right)^{0.9}\right]^2\right\} \\ &= \underline{82\ \mathrm{m}}\end{aligned}$$

This much simpler method provides the same value as that found using the Moody diagram.

Turbulent Flow in a Pipe

A pressure drop of 700 kPa is measured over a 300-m length of horizontal, 100-mm-diameter wrought iron pipe that transports oil ($S = 0.9$, $\nu = 10^{-5}$ m²/s). Calculate the flow rate using (a) the Moody diagram, and (b) the alternate method.

Solution

(a) The relative roughness is

$$\frac{e}{D} = \frac{0.046}{100} = 0.00046$$

Assuming that the flow is completely turbulent (Re is not needed), the Moody diagram gives

$$f = 0.0165$$

The head loss is found to be

$$h_L = \frac{\Delta p}{\gamma_{\text{oil}}} = \frac{700\,000 \text{ N/m}^2}{9800 \text{ N/m}^3 \times 0.9} = 79.4 \text{ m}$$

The velocity is calculated from Eq. 7.6.23 to be

$$V = \left(\frac{2gDh_L}{f\,L}\right)^{1/2} = \left(\frac{2 \times 9.8 \text{ m/s}^2 \times 0.1 \text{ m} \times 79.4 \text{ m}}{0.0165 \times 300 \text{ m}}\right)^{1/2} = 5.61 \text{ m/s}$$

This provides us with a Reynolds number of

$$\text{Re} = \frac{VD}{\nu} = \frac{5.61 \text{ m/s} \times 0.1 \text{ m}}{10^{-5} \text{ m}^2/\text{s}} = 5.61 \times 10^4$$

Using this Reynolds number and $e/D = 0.00046$, the Moody diagram gives the friction factor as

$$f = 0.023$$

This corrects the original value for f. The velocity is recalculated to be

$$V = \left(\frac{2 \times 9.8 \times 0.1 \times 79.4}{0.023 \times 300}\right)^{1/2} = 4.75 \text{ m/s}$$

Turbulent Flow in a Pipe

The Reynolds number is then

$$\mathrm{Re} = \frac{4.75 \times 0.1}{10^{-5}} = 4.75 \times 10^4$$

From the Moody diagram $f = 0.023$ appears to be satisfactory. Thus the flow rate is

$$Q = VA = 4.75 \times \pi \times 0.05^2 = \underline{0.037\ \mathrm{m^3/s}}$$

Only two significant numbers are given since f is known to at most two significant numbers.
(b) The alternative method for this Category 2 problem uses the explicit relationship (7.6.30). We can directly calculate Q to be

$$Q = -0.965\left(\frac{9.8 \times 0.1^5 \times 79.4}{300}\right)^{0.5} \ln\left[\frac{0.00046}{3.7} + \left(\frac{3.17 \times 10^{-10} \times 300}{9.8 \times 0.1^3 \times 79.4}\right)^{0.5}\right]$$

$$= -0.965 \times 5.096 \times 10^{-3} \times (-7.655) = \underline{0.038\ \mathrm{m^3/s}}$$

This much simpler method produces a value essentially the same as that obtained using the Moody diagram.

Turbulent Flow in a Pipe

Drawn tubing of what diameter should be selected to transport 0.002 m^3/s of 20°C water over a 400-m length so that the head loss does not exceed 30 m? (a) Use the Moody diagram and (b) the alternative method.

Solution

(a) In this problem we do not know D. Thus, a trial-and-error solution is anticipated. The average velocity is related to D by

$$V = \frac{Q}{A} = \frac{0.002}{\pi D^2/4} = \frac{0.00255}{D^2}$$

The friction factor and D are related as follows:

$$h_L = f\frac{L}{D}\frac{V^2}{2g}$$

$$30 = f\frac{400}{D}\frac{(0.00255/D^2)^2}{2\times 9.8}$$

$$\therefore D^5 = 4.42\times 10^{-6} f$$

The Reynolds number is

$$\text{Re} = \frac{VD}{\nu} = \frac{0.00255D}{D^2\times 10^{-6}} = \frac{2550}{D}$$

Now, let us simply guess a value for f and check with the relations above and the Moody diagram. The first guess is $f = 0.03$, and the correction is listed in the following table. Note: the second guess is the value for f found from the calculations of the first guess.

f	D(m)	Re	e/D	f(Figure 7.13)
0.03	0.0421	6.06×10^4	0.000036	0.02
0.02	0.0388	6.57×10^4	0.000039	0.02

Turbulent Flow in a Pipe

The value of $f = 0.02$ is acceptable, yielding a diameter of 38.8 mm. Since this diameter would undoubtedly not be standard, a diameter of

$$D = \underline{40\text{ mm}}$$

would be the tube size selected. This tube would have a head loss less than the limit of $h_L = 30$ m imposed in the problem statement. Any larger-diameter tube would also satisfy this criterion but would be more costly, so it should not be selected.

(b) The alternative method for this Category 3 problem uses the explicit relationship (7.6.31). We can directly calculate D to be

$$D = 0.66\left[(1.5\times10^{-6})^{1.25}\left(\frac{400\times0.002^2}{9.81\times30}\right)^{4.75} + 10^{-6}\times0.002^{9.4}\left(\frac{400}{9.81\times30}\right)^{5.2}\right]^{0.04}$$

$$= 0.66[5.163\times10^{-33} + 2.102\times10^{-31}]^{0.04} = \underline{0.039\text{ m}}$$

Hence $D = 40$ mm would be the tube size selected. This is the same tube size as that selected using the Moody diagram.

Turbulent Flow in a Pipe

Losses in Noncircular Conduits

- Can approximate for conduits with noncircular cross sections:
  - Using hydraulic radius R

$$R = \frac{A}{P}$$

A: Cross-sectional area
P: Wetted perimeter→ **Perimeter where the fluid is in contact with the solid boundary**

- E.g., for a circular pipe:
  - Hydraulic radius $R = r_o/2$

$$\mathrm{Re} = \frac{4VR}{\nu} \qquad \text{relative roughness} = \frac{e}{4R}$$

- The head-loss then becomes:

$$h_L = f\frac{L}{4R}\frac{V^2}{2g}$$

Turbulent Flow in a Pipe

Air at standard conditions is to be transported through 500 m of a smooth, horizontal, 300 mm × 200 mm rectangular duct at a flow rate of 0.24 m^3/s. Calculate the pressure drop.

Solution

The hydraulic radius is

$$R = \frac{A}{P} = \frac{0.3 \times 0.2}{(0.3 + 0.2) \times 2} = 0.06 \text{ m}$$

The average velocity is

$$V = \frac{Q}{A} = \frac{0.24}{0.3 \times 0.2} = 4.0 \text{ m/s}$$

This gives a Reynolds number of

$$\text{Re} = \frac{4VR}{\nu} = \frac{4 \times 4 \times 0.06}{1.5 \times 10^{-5}} = 6.4 \times 10^4$$

Using the smooth pipe curve of the Moody diagram, there results

$$f = 0.0196$$

Hence,

$$h_L = f\frac{L}{4R}\frac{V^2}{2g} = 0.0196\frac{500 \text{ m}}{4 \times 0.06 \text{ m}}\frac{4^2 \text{ m}^2/\text{s}^2}{2 \times 9.8 \text{ m/s}^2} = 33.3 \text{ m}$$

The pressure drop is

$$\Delta p = \rho g h_L = 1.23 \times 9.8 \times 33.3 = \underline{402 \text{ Pa}}$$

Turbulent Flow in a Pipe

Minor Losses in Pipe Flow

- Sometimes *minor losses* (from fittings that cause additional losses) can exceed frictional losses.
- Expressed in terms of a loss coefficient K.

$$h_L = K \frac{V^2}{2g}$$

- K can be determined experimentally.
  - If there is an expansion from area A_1 to area A_2:

$$h_L = \left(1 - \frac{A_1}{A_2}\right)^2 \frac{V_1^2}{2g}$$

  - For a sudden expansion in area:

$$K = \left(1 - \frac{A_1}{A_2}\right)^2$$

Turbulent Flow in a Pipe

Minor Losses in Pipe Flow

- A loss coefficient can be expressed as an ***equivalent length L_e*** of pipe*:*

$$L_e = K\frac{D}{f}$$

- For long segments of pipe, minor losses can usually be neglected.

Turbulent Flow in a Pipe

If the flow rate through a 100-mm-diameter wrought iron pipe (Figure E7.15) is 0.04 m^3/s, find the difference in elevation H of the two reservoirs.

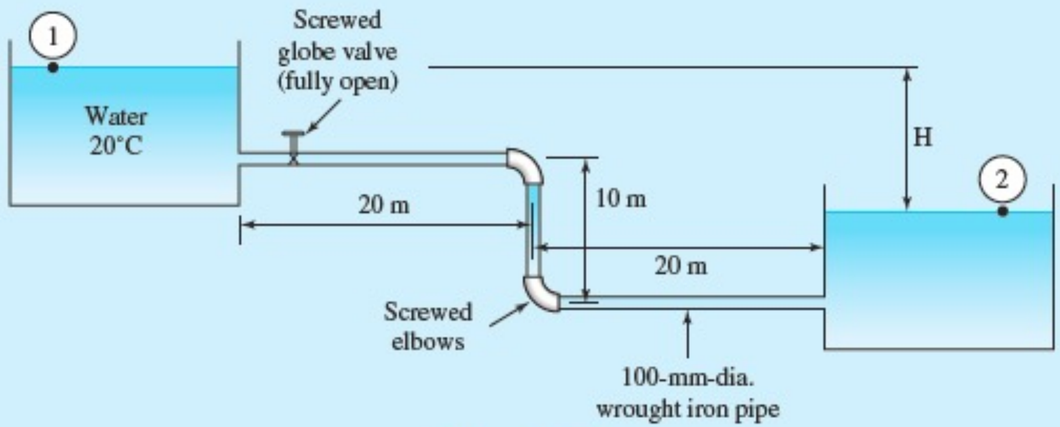


Figure E7.15

Solution

The energy equation written for a control volume that contains the two reservoir surfaces (see Eq. 4.5.17), where $V_1 = V_2 = 0$ and $p_1 = p_2 = 0$, is

$$0 = z_2 - z_1 + h_L$$

Thus, letting $z_1 - z_2 = H$, we have

$$H = (K_{\text{entrance}} + K_{\text{valve}} + 2K_{\text{elbow}} + K_{\text{exit}})\frac{V^2}{2g} + f\frac{L}{D}\frac{V^2}{2g}$$

Turbulent Flow in a Pipe

The average velocity, Reynolds number, and relative roughness are

$$V = \frac{Q}{A} = \frac{0.04}{\pi \times 0.05^2} = 5.09 \text{ m/s}$$

$$\text{Re} = \frac{VD}{\nu} = \frac{5.09 \times 0.1}{10^{-6}} = 5.09 \times 10^5$$

$$\frac{e}{D} = \frac{0.046}{100} = 0.00046$$

From the Moody diagram we find that

$$f = 0.0173$$

Using the loss coefficients from Table 7.2 for an entrance, a globe valve, screwed 10-cm-diameter standard elbows, and an exit there results

$$H = (0.5 + 5.7 + 2 \times 0.64 + 1.0)\frac{5.09^2}{2 \times 9.8} + 0.0173\frac{50}{0.1}\frac{5.09^2}{2 \times 9.8}$$

$$= 11.2 + 11.4 = \underline{22.6 \text{ m}}$$

Note: The minor losses are about equal to the frictional losses as expected, since there are five minor loss elements in 500 diameters of pipe length.

Turbulent Flow in a Pipe

Approximate the loss coefficient for the sudden contraction $A_1/A_2 = 2$ by neglecting the losses in the contracting portion up to the vena contracta and assuming that all the losses occur in the expansion from the vena contracta to A_2 (see Figure 7.16). Compare with that given in Table 7.2.

Solution

The head loss from the vena contracta to area A_2 is (see Table 7.2, sudden enlargement)

$$h_L = \left(1 - \frac{A_c}{A_2}\right)^2 \frac{V_c^2}{2g}$$

Continuity allows us to write (V_c is the velocity at the area A_c)

$$V_c = \frac{A_2}{A_c} V_2$$

Thus, the head loss based on V_2 is

$$h_L = \left(1 - \frac{A_c}{A_2}\right)^2 \left(\frac{A_2}{A_c}\right)^2 \frac{V_2^2}{2g}$$

so the loss coefficient of Eq. 7.6.35 is

$$K = \left(1 - \frac{A_c}{A_2}\right)^2 \left(\frac{A_2}{A_c}\right)^2$$

Using the expression of C_c given in Figure 7.16, we have

$$\frac{A_c}{A_2} = C_c = 0.62 + 0.38\left(\frac{1}{2}\right)^3 = 0.67$$

Finally,

$$K = (1 - 0.67)^2 \frac{1}{0.67^2} = \underline{0.24}$$

This compares favorably with the value of 0.25 given in Table 7.2.

Turbulent Flow in a Pipe

Hydraulic and Energy Grade Lines

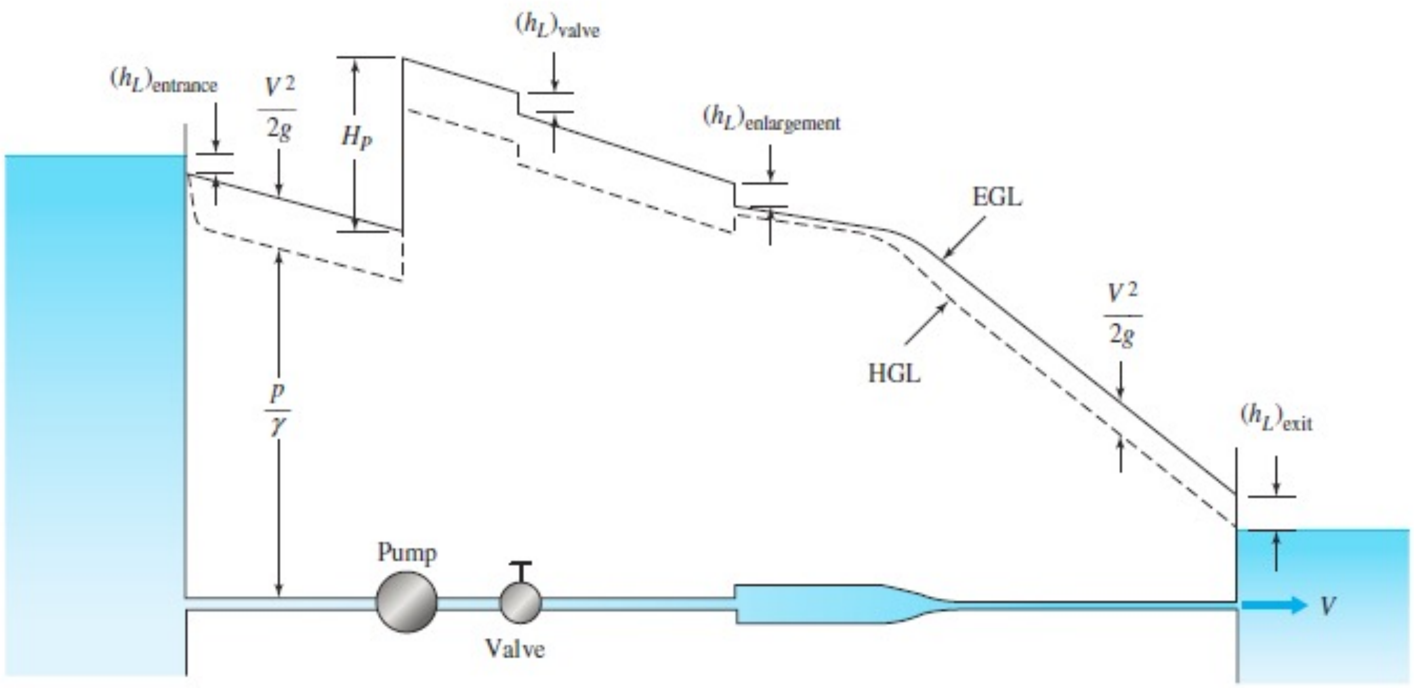
- Energy equation is written in a form where the terms have dimensions of length:

$$-\frac{\dot{W}_s}{\dot{m}g} - \frac{V_2^2 - V_1^2}{2g} + \frac{p_2 - p_1}{\gamma} + z_2 - z_1 + h_L$$

- The Hydraulic and Energy grade lines for piping systems can hence be defined:
  - **Hydraulic Grade Line (HGL):** Located a distance p/γ above the center of the pipe.
  - **Energy Grade Line (EFL):** Located a distance $V^2/2g$ above the HGL.

Turbulent Flow in a Pipe

Hydraulic and Energy Grade Lines



Hydraulic grade line (HGL) and energy grade line (EGL) for a piping system.

Turbulent Flow in a Pipe

Hydraulic and Energy Grade Lines: NOTES

- As V→0, HGL and EGL approach each other. (For a reservoir, they are identical and lie on the surface.)
- EGL and HGL slope downward in the flow direction (due to head loss).
    - Greater the loss per unit length, greater the slope.
- HGL and EGL suddenly change when a loss occurs due to sudden geometry changes.
    - Jumps when useful energy is added (pump).
    - Drops when useful energy is extracted (turbine).
- If the HGL passes through the centerline of the pipe→ pressure is zero.
    - If the pipe is above the HGL→ vacuum condition.

Turbulent Flow in a Pipe

Water at 20°C flows between two reservoirs at the rate of 0.06 m^3/s as shown in Figure E7.17. Sketch the HGL and the EGL. What is the minimum diameter D_B allowed to avoid the occurrence of cavitation?

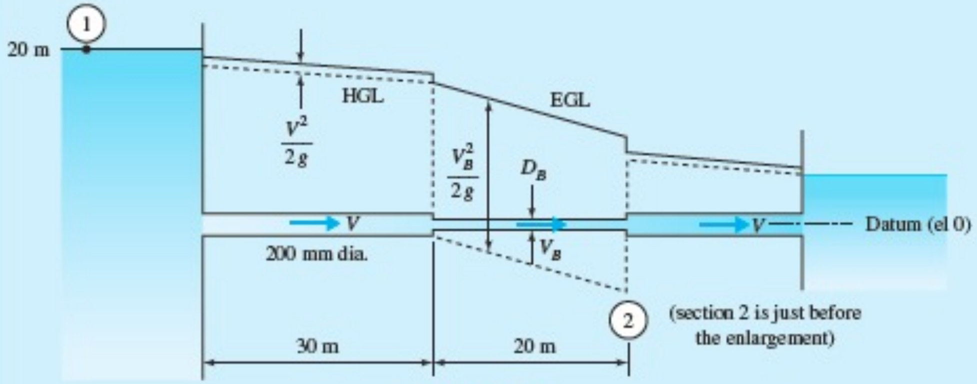


Figure E7.17

Solution

The EGL and the HGL are sketched on the figure, including sudden changes at the entrance, contraction, enlargement, and the exit. Note the large velocity head (the difference between the EGL and the HGL) in the smaller pipe because of the high velocity. The velocity, Reynolds number, and relative roughness in the 20-cm-diameter pipe are calculated to be

$$V = \frac{Q}{A} = \frac{0.06}{\pi \times 0.20^2/4} = 1.91 \text{ m/s}$$

$$\text{Re} = \frac{VD}{\nu} = \frac{1.91 \times 0.2}{10^{-6}} = 3.8 \times 10^5$$

$$\frac{e}{D} = \frac{0.26}{200} = 0.0013$$

Turbulent Flow in a Pipe

Thus $f = 0.022$ from Figure 7.13. The velocity, Reynolds number, and relative roughness in the smaller pipe are

$$V_B = \frac{0.06}{\pi D_B^2/4} = \frac{0.0764}{D_B^2}$$

$$\mathrm{Re}_B = \frac{0.0764 \times D_B}{D_B^2 \times 10^{-6}} = \frac{76\,400}{D_B}$$

$$\frac{e}{D_B} = \frac{0.00026}{D_B}$$

The minimum possible diameter is established by recognizing that the water vapor pressure (2450 Pa absolute) at 20°C is the minimum allowable pressure. Since the distance between the pipe and the HGL is an indication of the pressure in the pipe, we can conclude that the minimum pressure will occur at section 2. Hence the energy equation applied between section 1, the reservoir surface, and section 2 gives

$$\cancelto{0}{\frac{V_1^2}{2g}} + \frac{p_1}{\gamma} + z_1 = \frac{V_B^2}{2g} + \frac{p_2}{\gamma} + \cancelto{0}{z_2} + K_{\text{ent}}\frac{V_A^2}{2g} + K_{\text{cont}}\frac{V_B^2}{2g} + f_A\frac{L_A}{D_A}\frac{V_A^2}{2g} + f_B\frac{L_B}{D_B}\frac{V_B^2}{2g}$$

where the subscript A refers to the 20-cm-diameter pipe. This simplifies, using absolute pressure, to

$$\frac{101\,000}{9810} + 20 = \frac{\left(0.0764/D_B^2\right)^2}{2 \times 9.81}\left(1 + 0.25 + f_B\frac{20}{D_B}\right) + \frac{2450}{9810}$$

$$+ \left(0.5 + 0.022\frac{30}{0.2}\right)\frac{1.91^2}{2 \times 9.81}$$

$$98\ 600 = \frac{1.25}{D_B^4} + f_B\frac{20}{D_B^5}$$

Turbulent Flow in a Pipe

where we have used $K_{ent} = 0.5$ and assumed that $K_{cont} = 0.25$. This requires a trial-and-error solution. The following illustrates the procedure.

Let $D_B = 0.1\,\text{m}$. Then $e/D_B = 0.0026$ and $\text{Re}_B = 7.6 \times 10^5$. Therefore, $f = 0.026$:

$$98\ 600 \overset{?}{=} 12\ 500 + 52\ 000$$

Let $D_B = 0.09\,\text{m}$. Then $e/D_B = 0.0029$ and $\text{Re}_B = 8.4 \times 10^5$. Therefore, $f = 0.027$:

$$98\ 600 \overset{?}{=} 19\ 000 + 91\ 000$$

We see that 0.1 m is too large and 0.09 m is too small. In fact, the value of 0.09 m is only slightly too small. Consequently, to be safe we must select the next larger pipe size of 0.1 m diameter. If there were a pipe size of 95 mm diameter, that could be selected. Assuming that that size is not available, we select

$$D_B = \underline{\underline{100\ \text{mm}}}$$

Note that the assumption of a 2:1 area ratio for the contraction is too small. It is actually 4:1. This would give $K_{cont} \simeq 0.4$. After a quick check we conclude that this value does not significantly influence the result.

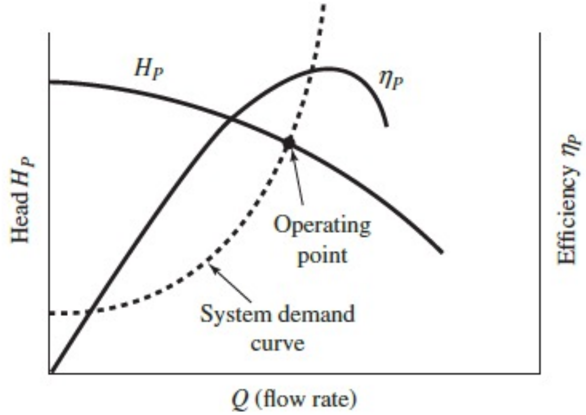
Turbulent Flow in a Pipe

Simple Pipe System with a Centrifugal Pump

- If the flow rate of the pump is not given→ Not straightforward.
  - The head produced by the pump and the efficiency depend on the discharge.
  - Need the characteristic curves of the pump.
  - Can relate flow rate Q, and pump head H_P.

$$H_P = c_1 + c_2 Q^2$$

System Demand Curve: Energy equation relating pump head to an unknown flow rate.



Pump characteristic curve (Intersection is the operating point of the system)

Turbulent Flow in a Pipe

Estimate the flow rate in the simple piping system of Figure E7.18a if the pump characteristic curves are as shown in Figure E7.18b. Also, find the pump power requirement.

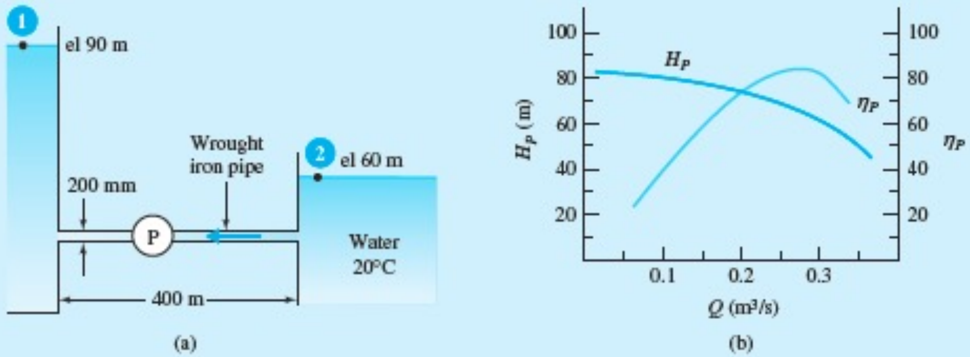


Figure E7.18

Solution

We will assume that the Reynolds number is sufficiently large that the flow is completely turbulent. So, using $e/D = 0.046/200 = 0.00023$, the friction factor from the Moody diagram is

$$f = 0.014$$

The energy equation (see Eq. 7.6.40), with $H_r = -\dot{W}_s/\dot{m}g$, applied between the two surfaces, yields

$$H_r = \frac{\cancelto{0}{V_2^2 - V_1^2}}{2g} + z_2 - z_1 + \frac{\cancelto{0}{p_2 - p_1}}{\gamma} + h_L$$

or

$$H_r = 90 - 60 + \left(K_{\text{entrance}} + K_{\text{exit}} + f\frac{L}{D}\right)\frac{V^2}{2g}$$

Turbulent Flow in a Pipe

$$= 30 + \left(0.5 + 1.0 + 0.014\frac{400}{0.2}\right)\frac{Q^2}{2 \times 9.8 \times [\pi \times 0.1^2]^2}$$

$$= 30 + 1520Q^2$$

This equation, the system demand curve, and the characteristic curve $H_r(Q)$ of the pump are now solved simultaneously by trial and error. Actually, the curve could be plotted on the same graph as the characteristic curve, and the point of intersection, the operating point, would provide Q. Try $Q = 0.2\ \text{m}^3/\text{s}$: $(H_r)_{\text{energy}} = 91\ \text{m}$, $(H_r)_{\text{char}} \cong 75\ \text{m}$. Try $Q = 0.15\ \text{m}^3/\text{s}$: $(H_r)_{\text{energy}} = 64\ \text{m}$, $(H_r)_{\text{char}} \cong 75\ \text{m}$. Try $Q = 0.17\ \text{m}^3/\text{s}$: $(H_r)_{\text{energy}} = 74\ \text{m}$, $(H_r)_{\text{char}} \cong 76\ \text{m}$. This is our solution. We have

$$Q = 0.17\ \text{m}^3/\text{s}$$

Check the Reynolds number: $\text{Re} = DQ/A\nu = 0.2 \times 0.17/(\pi \times 0.1^2 \times 10^{-6}) = 1.08 \times 10^6$. This is sufficiently large, but marginally so.

The power requirement of the pump is given by Eq. 4.5.26:

$$\dot{W}_r = \frac{Q\gamma H_r}{\eta_r}$$

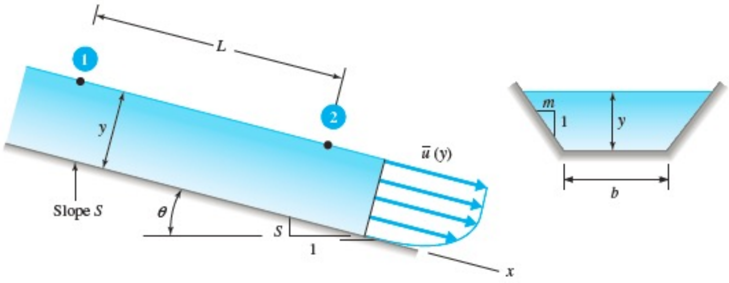
$$= \frac{0.17\ \text{m}^3/\text{s} \times 9800\ \text{N/m}^3 \times 75\ \text{m}}{0.65} = 198\ 000\ \text{W} \quad \text{or} \quad 198\ \text{kW}$$

where the efficiency $\eta_r = 0.65$ is found from the characteristic curve at $Q = 0.17\ \text{m}^3/\text{s}$. *Note:* Since $L/D > 1000$, minor losses due to the entrance and exit could have been neglected.

Uniform Turbulent Flow in Open Channels

- Steady, uniform flow in an open channel can be understood using the Darcy-Weisbach relation.
  - Uniform flow in an open, rough channel can be analyzed using the energy equation.

$$0 = \frac{V_2^2 - \cancel{V_1^2}}{\cancel{2g}} + \frac{P_2 - \cancel{P_1}}{\cancel{\gamma}} + z_2 - z_1 + h_L$$



Uniform flow in an open channel.

- Hence the head loss is simply:

$$\begin{aligned} h_L &= z_1 - z_2 \\ &= L\sin\theta = LS \end{aligned}$$

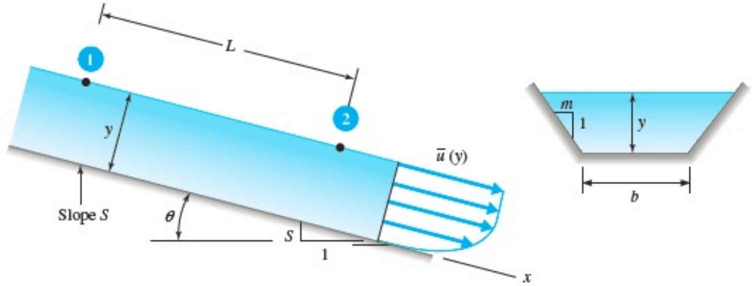
L: Length of the channel
S: Slope of the channel

- The Darcy-Weisbach equation for this head-loss is:

$$LS = f\frac{L}{4R}\frac{V^2}{2g} \qquad \therefore RS = \frac{f}{8g}V^2$$

R: Hydraulic radius

Uniform Turbulent Flow in Open Channels



Uniform flow in an open channel.

$$LS = f\frac{L}{4R}\frac{V^2}{2g} \qquad \therefore RS = \frac{f}{8g}V^2$$

- For large open channels (having large Reynolds numbers), the friction factor is in the turbulent region.

$$V = C\sqrt{RS}$$

- C = Chezy coefficient (dimensional constant)

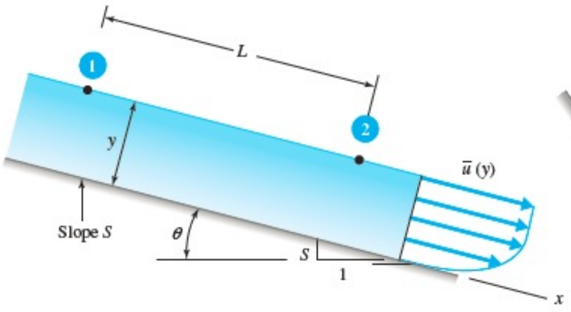
$$C = \frac{c_1}{n}R^{1/6}$$

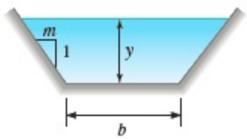
- C = Channel roughness
- R = Hydraulic radius
- c_1 = 1.0 (SI)

- n: Dimensionless constant related to the wall roughness (Manning n)

Wall material	*Manning n*
Planed wood	0.012
Unplaned wood	0.013
Finished concrete	0.012
Unfinished concrete	0.014
Sewer pipe	0.013
Brick	0.016
Cast iron, wrought iron	0.015
Concrete pipe	0.015
Riveted steel	0.017
Earth, straight	0.022
Corrugated metal flumes	0.025
Rubble	0.03
Earth with stones and weeds	0.035
Mountain streams	0.05

Uniform Turbulent Flow in Open Channels





$$LS = f\frac{L}{4R}\frac{V^2}{2g} \qquad \therefore RS = \frac{f}{8g}V^2$$

Uniform flow in an open channel.

- The flow-rate in an open channel is found using the **Chezy-Manning equation**.

$$Q = \frac{c_1}{n} A R^{2/3} S^{1/2}$$

$c_1 = 1.0$ for SI units

Note: Equation is usually used for rough-walled channels.

Wall material	*Manning n*
Planed wood	0.012
Unplaned wood	0.013
Finished concrete	0.012
Unfinished concrete	0.014
Sewer pipe	0.013
Brick	0.016
Cast iron, wrought iron	0.015
Concrete pipe	0.015
Riveted steel	0.017
Earth, straight	0.022
Corrugated metal flumes	0.025
Rubble	0.03
Earth with stones and weeds	0.035
Mountain streams	0.05

Uniform Turbulent Flow in Open Channels

The depth of water at 16°C flowing in a 3.6-m-wide rectangular, finished concrete channel is measured to be 1.2 m. The slope is measured to be 0.0016. Estimate the flow rate using (a) the Chezy-Manning equation and (b) the Darcy-Weisbach equation.

Solution

The hydraulic radius is calculated to be

$$R = \frac{A}{P} = \frac{yb}{2y+b} = \frac{1.2 \times 3.6}{2 \times 1.2 + 3.6} = 0.72\,\text{m}$$

(a) Using the Chezy-Manning equation, with $n = 0.012$ from Table 7.3 and $c = 1$, we have

$$\begin{aligned} Q &= \frac{1}{n} A R^{2/3} S^{1/2} \\ &= \frac{1\,\text{m}^{1/3}/\text{s}}{0.012} \times (1.2 \times 3.6)\,\text{m}^2 \times (0.72)^{2/3}\,\text{m}^{2/3} \times 0.0016^{1/2} = \underline{11.57\,\text{m}^3/\text{s}} \end{aligned}$$

(b) The relative roughness is, using a low value $e = 0.00045$ m (it is finished concrete) shown on the Moody diagram:

$$\frac{e}{4R} = \frac{0.00045}{4 \times 0.72} = 0.00016$$

Assuming a completely turbulent flow, the Moody diagram gives the friction factor as

$$f = 0.013$$

Uniform Turbulent Flow in Open Channels

The Darcy-Weisbach equation (7.7.3) then yields the velocity as follows:

$$\therefore V = \left(\frac{8RgS}{f}\right)^{1/2}$$

$$= \left(\frac{8 \times 0.72\,\text{m} \times 9.81\,\text{m/s}^2 \times 0.0016}{0.013}\right)^{1/2} = 2.64\,\text{m/s}$$

The flow rate is calculated as

$$Q = VA = 2.64 \times 1.2 \times 3.6 = \underline{11.4\ \text{m}^3/\text{s}}$$

These two values are within 2%, an acceptable engineering tolerance for this type of problem. That found using the Moody diagram is considered to be more accurate, however.

Uniform Turbulent Flow in Open Channels

A 1.0-m-diameter concrete pipe transports 20°C water at a depth of 0.4m. If the slope is 0.001, find the flow rate using (a) the Chezy-Manning equation and (b) the Darcy-Weisbach equation.

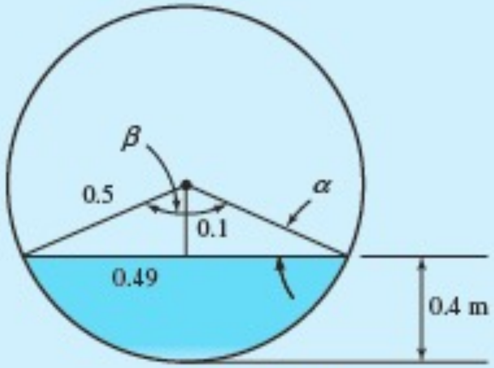


Figure E7.20

Solution

From the sketch of the pipe in Figure E7.20 the following are calculated:

$$\alpha = \sin^{-1}\frac{0.1}{0.5} = 11.54°$$

$$\therefore \beta = 180 - 2 \times 11.54 = 156.9°$$

$$\therefore A = \pi \times 0.5^2 \times \frac{156.9}{360} - 0.49 \times 0.1 = 0.2933\,\text{m}^2$$

$$P = 2\pi \times 0.5 \times \frac{156.9}{360} = 1.369\,\text{m}$$

Uniform Turbulent Flow in Open Channels

The hydraulic radius is found, using the above calculations, to be

$$R = \frac{A}{P} = \frac{0.2933}{1.369} = 0.2142\,\text{m}$$

(a) The Chezy-Manning equation yields, with n from Table 7.3 and $c_1 = 1.0\ \text{m}^{1/3}/\text{s}$,

$$Q = \frac{1.0}{n} AR^{2/3}S^{1/2} = \frac{1.0}{0.015} \times 0.2933 \times 0.2142^{2/3} \times 0.001^{1/2} = \underline{0.22\ \text{m}^3/\text{s}}$$

(b) The relative roughness is, using a relatively rough value for concrete pipe from Figure 7.13, as suggested by Table 7.3 of $e = 20$ mm,

$$\frac{e}{4R} = \frac{2}{4 \times 214.2} = 0.0023$$

Assuming completely turbulent flow, the Moody diagram yields

$$f = 0.025$$

The Darcy-Weisbach equation (7.7.3) then gives the following:

$$\therefore V = \left(\frac{8RgS}{f}\right)^{1/2} = \left(\frac{8 \times 0.2142 \times 9.81 \times 0.001}{0.025}\right)^{½} = 0.820\,\text{m/s}$$

The flow rate is

$$Q = VA = 0.820 \times 0.2933 = \underline{0.24\ \text{m}^3/\text{s}}$$

This is within 8% of the result above, an acceptable tolerance for this type of problem. The second method, which is more difficult to apply, is considered to be more accurate, however.

Summary

- Laminar entrance lengths for a pipe and wide channel are:

$$\frac{L_E}{D} = 0.065 \text{ Re} \qquad \frac{L_E}{h} = 0.04 \text{ Re}$$

- For high Reynolds-number turbulent pipe flow, the entrance length is:

$$\frac{L_E}{D} = 120$$

- For laminar flow in a pipe and a wide channel, the pressure drop and friction factor are:

a: Channel height

$$\Delta p = \frac{8\mu VL}{r_0^2} \qquad f = \frac{64}{\text{Re}} \qquad \text{pipe}$$

$$\Delta p = \frac{12\mu VL}{a^2} \qquad f = \frac{48}{\text{Re}} \qquad \text{channel}$$

Summary

- The torque required to rotate an inner cylinder with the outer cylinder fixed is:

$$T = \frac{4\pi\mu r_1^2 r_2^2 L\omega_1}{r_2^2 - r_1^2}$$

- The head loss in a turbulent flow is simply:

$$h_L = f\frac{L}{D}\frac{V^2}{2g}$$

f: Found using a Moody diagram

- Minor losses are included using loss coefficients, K:

$$h_L = K\frac{V^2}{2g}$$

- The flow rate in an open channel is estimated by:

$$Q = \frac{c_1}{n}AR^{2/3}S^{1/2} \qquad c_1 = 1.0\ \mathrm{m^{1/3}/s}$$